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On stringy AdS$_5 \times S^5$ and higher spin holography

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Abstract: We derive the spectrum of Kaluza-Klein descendants of string excitations on AdS$_5 \times S^5$. String states are organized in long multiplets of the AdS supergroup SU$(2,2|4)$ with a rich pattern of shortenings at the higher spin enhancement point $\lambda = 0$. The string states holographically dual to the higher spin currents of SYM theory in the strict zero coupling limit are identified together with the corresponding Goldstone particles responsible for the Higgsing of the higher spin symmetry at $\lambda \neq 0$. Exploiting higher spin symmetry we propose a very simple yet effective mass formula and establish a one-to-one correspondence between the complete spectrum of $\Delta_0 \leq 4$ string states and relevant/marginal single-trace deformations in $\mathcal{N} = 4$ SYM theory at large N. To this end, we describe how to efficiently enumerate scaling operators in 'free' YM theory, with the inclusion of fermionic 'letters', by resorting to Polya theory. Comparison between the spectra of $\frac{1}{4}$-BPS states is also presented. Finally, we discuss how to organize the spectrum of $\mathcal{N} = 4$ SYM theory in SU$(2,2|4)$ supermultiplets by means of some kind of 'Eratostene's sieve'.

Keywords: Superstrings and Heterotic Strings, Superstring Vacua, AdS-CFT and dS-CFT Correspondence, Space-Time Symmetries.

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1. Introduction and summary

The quest for a string description of gauge theories has a long history starting from the pioneering work of A. Polyakov [1] and G. 't Hooft [2]. These ideas have recently found a concrete realization in the remarkable proposal by J. Maldacena [3] for an exact correspondence between type-IIB superstring on $\mathrm{AdS}_5 \times S^5$ with N RR 5-form fluxes and $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory with N colors. At large N and fixed $\lambda = g^2_{\mathrm{YM}} N$, non-planar diagrams corresponding to string loops are suppressed but superstring quantization in the presence of RR backgrounds is poorly understood [4, 5], except possibly for pp-waves [6, 7] that describe the so-called BMN limit [7]. For this reason one has to resort to an effective low-energy supergravity description, valid at weak curvature (large radius) $R \gg \sqrt{\alpha'}$, that only captures the strong coupling regime $\lambda \gg 1$ of the gauge theory. As R is reduced towards its minimal value, $R \approx \sqrt{\alpha'}$, the supergravity description breaks down and stringy physics becomes relevant. On the SYM side, the theory is driven to its perturbative regime $\lambda \ll 1$, ending at the highly symmetric point of vanishing coupling. The dynamics at this point is highly constrained if not frozen by the presence of infinitely many higher spin symmetries, i.e. by an infinite set of conserved currents of arbitrarily high spins. Pushing Maldacena duality to its limit, one may expect a duality to a bulk theory with an infinite number of massless higher spin (HS) particles on AdS.[1]

In flat spacetime a non-trivial dynamics in the analogous zero tension limit of superstring theory seems to be ruled out by the theorem of S. Coleman and J. Mandula. The assumptions of this theorem are violated by the presence of a non-trivial cosmological constant, and one may ask whether there can be an interacting theory of higher spins (HS) on AdS. The answer seems to be affirmative. The interactions that correspond to joining and breaking of strings should presumably be governed by $g_s \approx 1/N$ in this limit [8].

This seems exactly what the doctor ordered since progress in constructing consistent theories for interacting fields of higher spin has for long been hampered by the assumption that flat spacetime would be a good starting point. Quite amazingly, on the contrary the best starting point seems to be AdS [9, 10]. In a remarkable series of papers M. Vasiliev and collaborators have developed a formalism that consistently describes interacting HS theories in $D = 4$ [11] and made some progress towards solving the same problem in $D = 5$ [12] or even in arbitrary dimension for totally symmetric tensors [13]. A hint on how the former case can find application comes from the recent proposal of I. Klebanov and A. Polyakov [14] that has been generalized by the work of several groups [15]–[19]. Free geometric equations for massless HS gauge fields in flat $D = 4$ spacetime have also been recently proposed [20].

In the search for consistent HS theories in $D = 5$, E. Sezgin and P. Sundell [8] have shown that superstring states belonging to the first Regge trajectory on AdS can be put in one to one correspondence with the physical states in the master fields of Vasiliev's theory that in turn encompass bilinear (twist two) composites on the boundary. The linearized bulk field equations assume the desired form. In particular, one finds the correct self-duality constraints emerging from the natural curvature constraints. The energy-spin relation turns out to be linear, $MR \approx s$, rather than quadratic $M^2 = (s-1)/\alpha'$ as for superstrings in flat spacetime. This linear behavior has been confirmed by studies of 'long' solitonic strings in AdS, as opposed to 'short' ones that behave as in flat space [21]. In the spirit of holography the massless higher spin gauge theory describes the dynamics of twist two operators with conformal dimensions saturating the AdS unitarity bound:

$$\Delta = s + 2\,. \tag{1.1}$$

Another very interesting aspect of the problem is what happens when the string tension is small but non-vanishing. Out of the infinite number of conserved HS currents only a handful are preserved, while the infinitely many remaining ones are violated. In conformal

[1]By masslessness in AdS we understand particles associated to gauge invariances.

field theory, and this is the case if one simply turns on the gauge coupling rather than relevant operators, violating the conservation of a current of any spin is a symptom of its acquiring an anomalous dimension $\Delta = 2 + s + \gamma$ [22]–[27]. Global symmetries on the boundary correspond to local symmetries in the bulk; thus the holographic counterpart of this is a unprecedented Higgs mechanism in which infinitely many massless HS particles eat infinitely many lower spin Goldstone particles and become massive. Ideally one would like to describe this "grande bouffe" at the fully non-linear level, in the hope that the coupling of the Goldstone master fields to Vasiliev's master fields, as well as the self-interaction of the latter, be fixed by higher spin symmetry. Less ambitiously one may ask whether the resulting 'mass shifts' to lowest order match the one-loop anomalous dimensions that have a strikingly simple form [28, 29]

$$\gamma^{(s)}_{\text{1-loop}}(\lambda) = \frac{\lambda}{2\pi^2} \sum_{k=1}^{s-2} \frac{1}{k} \tag{1.2}$$

where s is the top spin in the (long) multiplet. In order to do that one may not forgo understanding the structure of the Goldstone master fields and their possible mixing with an infinite tower of Kaluza-Klein (KK) master fields that unavoidably appear when the HS theory is viewed as a consistent truncation of type-IIB superstring compactification on S^5.

The aim of the present paper is to set the basis for a systematic study of these issues. We will focus on the spectrum of the theory, leaving the issue of interactions to future investigations. KK reductions of supergravities on $\mathrm{AdS}_p \times S^q$ spaces have been studied in various contexts [30, 31, 32]. Here we apply these techniques in order to determine the spectrum of KK descendants of higher string excitations on $\mathrm{AdS}_5 \times S^5$.

Before entering the technical description of our results it is worth spending some words to explain our general philosophy. There are two choices for maximally supersymmetric vacua of string theory in $D = 5$. They correspond to compactifications of type-IIB theory on T^5 and on S^5. At low energies, the former leads to $\mathcal{N} = 8$ 'rigid' (or rather 'ungauged') supergravity, while the latter is believed to be governed by $\mathcal{N} = 8$ SO(6) gauged supergravity. The two vacua can be 'continuously' related by formally turning on and off the gauge coupling constant in $D = 5$. In both cases, it should be consistent at tree level to neglect KK descendants as well as various p-brane wrappings and only retain fundamental string excitations.[2] Agreement between the two pictures then requires that the ground 'floor' in the towers of KK descendants of type-IIB string excitations on $\mathrm{AdS}_5 \times S^5$ accommodate the spectrum of the rigid theory when the latter is properly rearranged in representations of the AdS supergroup. This will be the starting point (or assumption) of our KK algorithm.

More precisely, to each string excitation on flat space we will associate a tower of KK descendants on S^5. Proceeding in this way one automatically ensures that the ground floors in the KK towers on S^5 account for the right number of superstring degrees of freedom in the rigid limit. The resulting theory can then be thought of as an SO(6) gauging of

[2] Despite our limited understanding of superstring quantization on $\mathrm{AdS}_5 \times S^5$ [4, 5], the absence of non-trivial cycles in S^5 should be taken as a positive indication in this respect.

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type-IIB superstring theory dimensionally reduced on T^5 much in the same way as $\mathcal{N}=8$ gauged supergravity can be realized as a gauging of type-IIB supergravity reduced on T^5.

Field equations for string fluctuations around AdS vacua are not available even at the linearized level except for 'massless states' [8]. As a result the dynamical information about scaling dimensions Δ will be missing in our initial analysis. Unlike the remaining $\mathrm{SO}(4)\times\mathrm{SO}(6)$ quantum numbers in the bosonic symmetry group that will be unambiguously determined by our KK algorithm, conformal dimensions are sensible to quantum corrections. In the planar limit, the spectrum of $\Delta=\Delta(\lambda)$ should be determined by consistent quantization of the superstring on $\mathrm{AdS}_5\times S^5$ [4, 5]. Despite some recent progress there seems to be a long way to go. Still, supersymmetry and higher spin symmetries can be exploited in order to get some insight into the general form of the spectrum of conformal dimensions Δ_0 at $\lambda=0$. The recent observations in [33]–[37] raise some hope of a possible direct evaluation of the spectrum in an effective worldsheet theory. We hope to explore these ideas in the near future.

The spectrum of KK descendants that we have derived provides us with a rather rich set of data on which general ideas about holography beyond the supergravity approximation can be quantitatively tested. Indeed as we shall see, the resulting spectrum captures most of the interesting features of HS AdS/CFT holography. In particular, the gauge multiplets realizing the expected $hs(2,2|4)$ higher spin symmetry appearing in $\mathcal{N}=4$ SYM at $\lambda=0$ [38, 9, 8] are identified in the string picture together with their associated Goldstone particles. Quite remarkably if not unexpectedly, given our rather bold assumptions, we will establish a one-to-one correspondence between the spectrum of $\Delta_0\leq 4$ superstring excitations and that of relevant and marginal single-trace primary operators of $\mathcal{N}=4$ SYM.[3] Conformal descendants are accounted for by the zero-modes of the string coordinates (4d 'momentum'). Mixing with multi-particle states corresponding to multi-trace operators is suppressed at large N [26, 27, 39] and will not be discussed any further. Nor shall we discuss the spectrum from the viewpoint of the light-cone gauge quantization [40].

The plan of the paper is as follows: in section 2 we review the construction of the massive string spectrum of type-IIB theory in flat spacetime. In section 3 we discuss our 'naive' KK reduction on $\mathrm{AdS}_5\times S^5$ and exploit HS symmetry at small radius. In section 5 we compare the resulting spectrum with that of the free $\mathcal{N}=4$ SYM theory. To this end we compute in section 4 the partition function of gauge-invariant (on-shell) single-trace operators using Polya theory [41] and generalizing A. Polyakov's results [42], by the inclusion of fermionic "letters". This section is self-contained and can be read independently of the rest of the paper. We also discuss superconformal transformations that are needed to identify 'HWS' (superconformal primaries) together with an alternative procedure based on 'Eratostene's (super)sieve'. In section 6 we draw some conclusions. Finally, in two appendices we collect some background material, a self-contained description of multiplet shortenings, largely based on the work of F. Dolan and H. Osborn [28], and several useful tables.

[3]The agreement seems to persist up to scaling dimension $\Delta<6$, including $\Delta=5.5$, where a fermionic superconformal primary first appears. We thank N. Beisert for sharing his insights on this point with us.

2. Type-II superstrings in flat space

We start by reviewing the spectrum of type-II superstrings in flat space [43] in the GS formulation wherein chiral string excitations are created by the raising modes α^I_{-n}, S^a_{-n} acting on the vacuum $|\mathcal{Q}_c\rangle$:

$$S^a_n|\mathcal{Q}_c\rangle = \alpha^I_n|\mathcal{Q}_c\rangle = 0 \qquad n > 0\,. \tag{2.1}$$

Here and below indices $I = 1, \ldots, \mathbf{8_v}$, $a = 1, \ldots, \mathbf{8_s}$, $\dot{a} = 1, \ldots, \mathbf{8_c}$ run over the vector, spinor left and spinor right representations of the SO(8) little Lorentz group. In addition we have introduced the compact notations:

$$\mathcal{Q}_s = \mathbf{8_v} + \mathbf{8_s} \qquad \mathcal{Q}_c = \mathbf{8_v} + \mathbf{8_c}\,, \tag{2.2}$$

to describe chiral worldsheet supermultiplets. The vacuum $|\mathcal{Q}_c\rangle$ is 2^4-fold degenerated as a result of the quantization of the eight fermionic zero modes S^a_0. For the first few string excitation levels one finds:

$$\begin{aligned}
\ell = 0 \quad & |\mathcal{Q}_c\rangle\,,\\
\ell = 1 \quad & (S^a_{-1} + \alpha^I_{-1})|\mathcal{Q}_c\rangle\,,\\
\ell = 2 \quad & (S^a_{-2} + \alpha^I_{-2})|\mathcal{Q}_c\rangle\,, \quad S^{[a}_{-1}S^{b]}_{-1}|\mathcal{Q}_c\rangle\,, \quad \alpha^{(I}_{-1}\alpha^{J)}_{-1}|\mathcal{Q}_c\rangle\,, \quad S^a_{-1}\alpha^I_{-1}|\mathcal{Q}_c\rangle\,,\\
\ell = 3 \quad & (S^a_{-3} + \alpha^I_{-3})|\mathcal{Q}_c\rangle\,, \quad (S^a_{-2} + \alpha^I_{-2})(S^a_{-1} + \alpha^I_{-1})|\mathcal{Q}_c\rangle\,, \quad S^{[a_1}_{-1}\ldots S^{a_3]}_{-1}|\mathcal{Q}_c\rangle\,,\\
& \alpha^{(I_1}_{-1}\ldots\alpha^{I_3)}_{-1}|\mathcal{Q}_c\rangle \quad S^{[a_1}_{-1}S^{a_2]}_{-1}\alpha^I_{-1}|\mathcal{Q}_c\rangle\,, \quad \alpha^{(I_1}_{-1}\alpha^{I_2)}_{-1}S^a_{-1}|\mathcal{Q}_c\rangle\,.
\end{aligned} \tag{2.3}$$

The physical spectrum of the type-IIB superstring is defined by tensoring two (left and right moving) identical chiral spectra subject to the level matching condition $N_L = N_R$. Type-IIA string states are given instead by tensoring chiral spectra with opposite chiralities. Decomposing into SO(8) representations the spectrum T_ℓ at string excitation level ℓ yields:

$$\begin{aligned}
T_0^{IIA,B} &= \mathcal{Q}_c\, Q_{c,s}\,,\\
T_1 &= \mathcal{Q}_c^2\mathcal{Q}_s^2\,,\\
T_2 &= \mathcal{Q}_c^2(\mathcal{Q}_s + \mathcal{Q}_s\cdot\mathcal{Q}_s)^2 = T_1 \times (\mathbf{1} + \mathbf{8_v})^2\,,\\
T_3 &= \mathcal{Q}_c^2(\mathcal{Q}_s + \mathcal{Q}_s^2 + \mathcal{Q}_s\cdot\mathcal{Q}_s\cdot\mathcal{Q}_s)^2 = T_1 \times (\mathbf{1} + \mathbf{8_v} + \mathbf{35_v} + \mathbf{8_s} + \mathbf{8_c})^2\,,
\end{aligned} \tag{2.4}$$

and so on. In the right hand sides of (2.4) we have used SO(8) group theory in such a way as to factor out T_1 that represents the contribution of a 'minimal' massive multiplet with exactly 2^{16} states. Thanks to the presence of the overall $\mathcal{Q}_c^2$ we need only factor out two $\mathcal{Q}_s$, each coming from rewriting dotted (graded symmetrized) products according to

$$\begin{aligned}
\mathcal{Q}_s\cdot\mathcal{Q}_s &\equiv 8_{(v}\times 8_{v)} + 8_{[s}\times 8_{s]} + 8_v\times 8_s = \mathbf{8_v}\times\mathcal{Q}_s\,,\\
\mathcal{Q}_s\cdot\mathcal{Q}_s\cdot\mathcal{Q}_s &\equiv 8_{(v}\times 8_v\times 8_{v)} + 8_{[s}\times 8_s\times 8_{s]} + 8_v\times 8_{[s}\times 8_{s]} + 8_s\times 8_{(v}\times 8_{v)}\\
&= (\mathbf{35}_v + \mathbf{8_c})\times\mathcal{Q}_s\,.
\end{aligned} \tag{2.5}$$

Clearly the spectrum of massive superstring excitations being non-chiral makes no distinction between type-IIA and type-IIB theories. This can be seen in (2.4) where beyond T_0, string excitations always come in left-right symmetric representations of SO(8). Consequently, the spectrum of $\ell \geq 1$ string excitations in (2.4) can be reorganized in terms of the SO(9) little Lorentz group of a massive particle in ten dimensions

$$
\begin{aligned}
T_1 &= ([2,0,0,0] + [0,0,1,0] + [1,0,0,1])^2 = (\mathbf{44} + \mathbf{84} + \mathbf{128})^2 \,, \\
T_\ell &= T_1 \times (\mathrm{vac}_\ell)^2 \,,
\end{aligned} \tag{2.6}
$$

with

$$
\begin{aligned}
\mathrm{vac}_1 &= [0,0,0,0] = \mathbf{1} \,, \\
\mathrm{vac}_2 &= [1,0,0,0] = \mathbf{9} \,, \\
\mathrm{vac}_3 &= [2,0,0,0] + [0,0,0,1] = \mathbf{44} + \mathbf{16} \,, \\
\mathrm{vac}_\ell &= [\ell-1,0,0,0] + \cdots \,.
\end{aligned} \tag{2.7}
$$

where $[a,b,c,d]$ are SO(9) Dynkin labels. Notice that fermionic HWS, corresponding to $\mathbf{44}\times\mathbf{16}$ and $\mathbf{16}\times\mathbf{44}$, first appear at level $\ell = 3$. As an illustration, we have listed the vac_ℓ for the lowest massive levels $\ell \leq 8$ in appendix B.

3. Type-IIB string on $\mathbf{AdS}_5 \times S^5$

To linear order in fluctuations around $\mathrm{AdS}_5 \times S^5$ and after diagonalization, the type-IIB field equations should boil down to a set of uncoupled free massive equations:

$$
\left(\nabla^2_{AdS_5\times S^5} - M^2_\Phi\right) \Phi_{\{\mu\}\,\{i\}} = 0 \,. \tag{3.1}
$$

The collective indices $\{\mu\} \in \mathcal{R}_{\mathrm{SO}(1,4)}$ and $\{i\} \in \mathcal{R}_{\mathrm{SO}(5)}$ label irreducible representations of the $\mathrm{SO}(1,4)\times\mathrm{SO}(5)$ subgroup of the SO(1,9) Lorentz group in $D = 10$, and run over the spectrum of type-IIB string excitations in flat space. As explained above, this guarantees the right behavior of the KK spectrum in the rigid limit. The form of (3.1) is fixed by Lorentz covariance, while the spectrum of "masses" M^2_Φ describing the coupling of a given field $\Phi_{\{\mu\}\,\{i\}}$ to the curvature and RR 5-form flux should be ideally determined by requiring BRS invariance of superstring propagation on $\mathrm{AdS}_5 \times S^5$.

The ten-dimensional field $\Phi_{\{\mu\}\,\{i\}}$ can be expanded in S^5-spherical harmonics:

$$
\Phi_{\{\mu\}\,\{i\}}(x,y) = \sum_{[k,p,q]} \mathcal{X}^{[kpq]}_{\{\mu\}}(x)\, \mathcal{Y}^{[kpq]}_{\{i\}}(y) \,, \tag{3.2}
$$

with x, y coordinates along AdS_5 and S^5 respectively. The sum runs over a set (to be determined below) of allowed representations of the S^5 isometry group[4] $\mathrm{SO}(6) \approx \mathrm{SU}(4)$ characterized by their SU(4) Dynkin labels $[k,p,q]$. Finally the (generalized) spherical

[4]Throughout, we will indistinguishly refer to this group as SO(6) or SU(4) and exclusively use SU(4) Dynkin labels.

harmonic functions $\mathcal{Y}^{[kpq]}_{\{i\}}(y)$ are eigenfunctions of the laplacian:

$$\nabla^2_{S^5}\,\mathcal{Y}^{[kpq]}_{\{i\}} = -\frac{1}{R^2}\left(C_2\left[\mathrm{SU}(4)\right] - C_2\left[\mathrm{SO}(5)\right]\right)\,\mathcal{Y}^{[kpq]}_{\{i\}}\,, \tag{3.3}$$

with $C_2[G]$ standing for the second Casimir of the group G.

The aim of this section is to derive the spectrum of harmonics $\mathcal{Y}^{[kpq]}_{\{i\}}(y)$ entering the expansion (3.2) using and extending the standard group theory techniques, see e.g. [31, 32] for details.

3.1 KK spectrum

Let us denote by $\mathcal{R}_{\mathrm{SO}(5)}$ a representation of SO(5) appearing in the decomposition of the spectrum of type-IIB string excitations on T^5 under $\mathrm{SO}(1,4)\times\mathrm{SO}(5)\subset\mathrm{SO}(1,9)$. Each SO(5) representation $\mathcal{R}_{\mathrm{SO}(5)}$ can be associated to a tower of KK descendants on S^5 belonging to representations of SO(6) that contain $\mathcal{R}_{\mathrm{SO}(5)}$ in the decomposition $\mathrm{SO}(5)\subset\mathrm{SO}(6)$. Denoting by $[m,n]$ the SO(5) Dynkin labels, the set of $[k,p,q]$ in the KK towers (3.2) is explicitly given by

$$\begin{aligned}\mathrm{KK}_{[m,n]} \equiv\; & \sum_{r=0}^{m}\sum_{s=0}^{n}\sum_{p=m-r}^{\infty}\,[r+s,\,p,\,r+n-s]\,+\\ & +\sum_{r=0}^{m-1}\sum_{s=0}^{n-1}\sum_{p=m-r-1}^{\infty}\,[r+s+1,\,p,\,r+n-s]\,.\end{aligned} \tag{3.4}$$

For instance, the KK towers arising from the lowest SO(5) representations are given by

$$\begin{aligned}\mathrm{KK}_{\mathbf{1}} &\equiv \mathrm{KK}_{[0,0]} = \sum_{p\geq 0}[0,p,0]\,,\\ \mathrm{KK}_{\mathbf{4}} &\equiv \mathrm{KK}_{[0,1]} = \sum_{p\geq 0}\left([0,p,1]+[1,p,0]\right)\\ \mathrm{KK}_{\mathbf{5}} &\equiv \mathrm{KK}_{[1,0]} = \sum_{p\geq 0}\left([0,p+1,0]+[1,p,1]\right)\,,\qquad \text{etc.}\end{aligned} \tag{3.5}$$

For our subsequent analysis, it is convenient to observe that, if the (in general reducible) SO(5) representation $\mathcal{R}_{\mathrm{SO}(5)}$ itself may be lifted to a representation $\widehat{\mathcal{R}}_{\mathrm{SO}(5)}$ of SO(6), its KK tower takes the very compact form

$$\mathrm{KK}_{\mathcal{R}_{\mathrm{SO}(5)}} = \sum_{n=0}^{\infty}[0,n,0]\times\widehat{\mathcal{R}}_{\mathrm{SO}(5)}\,, \tag{3.6}$$

where the r.h.s. now describes a tensor product in SU(4). It is obvious, that the states on the r.h.s. contain the states of $\mathcal{R}_{\mathrm{SO}(5)}$ under the breaking of SO(6) to SO(5). With some more effort and using the explicit form of the KK tower (3.4) the reader may convince himself that these and only these representations satisfy the embedding condition (3.4).

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As an illustration, one can explicitly verify this for the first few SO(5) representations whose KK towers are given above:

$$\begin{aligned}
\mathrm{KK}_{\mathbf{1}} &= \sum_{n=0} [0,n,0] \times \mathbf{1}\,, \\
\mathrm{KK}_{\mathbf{1+5}} &= \sum_{n=0} [0,n,0] \times \mathbf{6}\,, \\
\mathrm{KK}_{\mathbf{4}} &= \sum_{n=0} [0,n,0] \times \mathbf{4_s} = \sum_{n=0} [0,n,0] \times \mathbf{4_c}\,,
\end{aligned} \tag{3.7}$$

where the l.h.s. refer to SU(4) tensor products.[5] The last example in (3.7) shows that the presentation of a given KK tower may not be unique if $\mathcal{R}_{\mathrm{SO}(5)}$ admits different lifts to SO(6). Nevertheless the full KK tower is unambiguously determined. Summarizing the spectrum of harmonics can be found by first lifting $\mathcal{R}_{\mathrm{SO}(5)}$ to SO(6) and then tensoring it with $[0n0]$.

Let us now apply this analysis to the massive string spectrum (2.6). These states have been organized according to the little group SO(9). Following the above algorithm, we decompose this spectrum under SO(5) $\times$ SO(4) and lift SO(5) to SO(6) according to (3.4). This uniquely yields the decomposition of the full spectrum of KK towers under SO(6) $\times$ SO(4), corresponding to the product of the S^5 isometry group and the subgroup SO(4) $\subset$ SO(4, 2) of AdS_5 isometries.

The string spectrum on $\mathrm{AdS}_5 \times S^5$ is organized in generically long multiplets $\mathcal{A}^{\Delta}_{[k,p,q](j,\bar{\jmath})}$ of the AdS_5 supergroup SU(2, 2|4). HWS are characterized by their $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ spins $(j, \bar{\jmath})$ and SU(4) Dynkin labels $[k, p, q]$, that determine the dimension of the supermultiplet to be $\dim(\mathcal{A}^{\Delta}_{[k,p,q](j,\bar{\jmath})}) = 2^{16} \times \dim[k,p,q]_{(j,\bar{\jmath})}$, and by their scaling dimension $\Delta = \Delta_0 + \gamma$ that satisfy unitarity lower bounds (C.6) in terms of the remaining quantum numbers. The SU(4) $\times$ SO(4) content of long supermultiplets[6] can be read off from

$$\mathcal{A}^{\Delta_0}_{[k,p,q](j,\bar{\jmath})} \equiv \hat{T}^{(2)}_1 \times [k,p,q]^{\Delta_0 - 2}_{(j,\bar{\jmath})}\,, \tag{3.8}$$

where $\hat{T}^{(2)}_1$ can be identified with the by-now-famous (long) Konishi multiplet [44, 45, 25], whose HWS is a scalar singlet of bare scaling dimension $\Delta_0 = 2$. Its 2^{16} components arise from the unconstrained action of 16 supercharges Q, i.e.

$$\hat{T}^{(2)}_1 = (1 + Q + Q \wedge Q + \cdots) \times [000]^2_{(0,0)}\,, \qquad Q = [100]^{\frac{1}{2}}_{(\frac{1}{2},0)} + [001]^{\frac{1}{2}}_{(0,\frac{1}{2})}\,. \tag{3.9}$$

The quantum numbers of $\hat{T}^{(2)}_1$ were displayed in tables 1–15 of [44]. For convenience of the reader we collect these states in table 1.[7] Given the conformal dimension of the HWS the dimensions of its (super)-descendants can be read off from (3.9) by assigning $\Delta_Q = \frac{1}{2}$ to the supercharges Q. The conformal dimension Δ of the HWS should be fixed by quantizing the superstring fluctuations around the AdS vacuum. In the limit where the HS symmetry is restored, a large fraction of the spectrum of dimensions $\Delta(\lambda = 0) = \Delta_0$ is expected to

[5] Recall that the representations $\mathbf{4_{s,c}}$, $\mathbf{6}$ carry SU(4) Dynkin labels [100], [001], and [010], respectively.

[6] Unless otherwise stated, we will always indicate the bare dimension Δ_0 as a superscript.

[7] An extra singlet $(\theta^4\bar{\theta}^8)_{\ell+6}$ has been included in order to correct a misprint in table 4 of [44].

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Δ	
2	$[0,0,0]_{(0,0)}$
$\frac{5}{2}$	$[0,0,1]_{(0,\frac12)} + [1,0,0]_{(\frac12,0)}$
3	$[0,0,0]_{(\frac12,\frac12)} + [0,0,2]_{(0,0)} + [0,1,0]_{(0,1)+(1,0)} + [1,0,1]_{(\frac12,\frac12)} + [2,0,0]_{(0,0)}$
$\frac{7}{2}$	$[0,0,1]_{(\frac12,0)+(\frac12,1)+(\frac32,0)} + [0,1,1]_{(0,\frac12)+(1,\frac12)} + [1,0,0]_{(0,\frac12)+(0,\frac32)+(1,\frac12)} + [1,0,2]_{(\frac12,0)}$ $+[1,1,0]_{(\frac12,0)+(\frac12,1)} + [2,0,1]_{(0,\frac12)}$
4	$[0,0,0]_{(0,0)+(0,2)+(1,1)+(2,0)} + [0,0,2]_{(\frac12,\frac12)+(\frac32,\frac12)} + [0,1,0]_{2(\frac12,\frac12)+(\frac12,\frac32)+(\frac32,\frac12)} + [2,0,2]_{(0,0)} + [2,1,0]_{(0,1)}$ $+[0,1,2]_{(1,0)} + [0,2,0]_{2(0,0)+(1,1)} + [1,0,1]_{(0,0)+2(0,1)+2(1,0)+(1,1)} + [1,1,1]_{2(\frac12,\frac12)} + [2,0,0]_{(\frac12,\frac12)+(\frac12,\frac32)}$
$\frac{9}{2}$	$[0,0,1]_{(0,\frac12)+(0,\frac32)+2(1,\frac12)+(1,\frac32)+(2,\frac12)} + [0,0,3]_{(\frac32,0)} + [0,1,1]_{3(\frac12,0)+2(\frac12,1)+(\frac32,0)+(\frac32,1)} + [0,2,1]_{(0,\frac12)+(1,\frac12)}$ $+[1,0,0]_{(\frac12,0)+2(\frac12,1)+(\frac12,2)+(\frac32,0)+(\frac32,1)} + [1,0,2]_{(0,\frac12)+2(1,\frac12)} + [1,1,0]_{3(0,\frac12)+(0,\frac32)+2(1,\frac12)+(1,\frac32)}$ $+[1,1,2]_{(\frac12,0)} + [1,2,0]_{(\frac12,0)+(\frac12,1)} + [2,0,1]_{(\frac12,0)+2(\frac12,1)} + [2,1,1]_{(0,\frac12)} + [3,0,0]_{(0,\frac32)}$
5	$[0,0,0]_{(\frac12,\frac12)+(\frac12,\frac32)+(\frac32,\frac12)+(\frac32,\frac32)} + [0,0,2]_{2(0,0)+(1,0)+2(1,1)+(2,0)} + [0,1,0]_{3(0,1)+3(1,0)+2(1,1)+(1,2)+(2,1)}$ $+[0,1,2]_{2(\frac12,\frac12)+(\frac32,\frac12)} + [0,2,0]_{3(\frac12,\frac12)+(\frac12,\frac32)+(\frac32,\frac12)} + [0,2,2]_{(0,0)} + [0,3,0]_{(0,1)+(1,0)}$ $+[1,0,3]_{(1,0)} + [1,1,1]_{2(0,0)+2(0,1)+2(1,0)+2(1,1)} + [1,2,1]_{(\frac12,\frac12)} + [2,0,0]_{2(0,0)+(0,1)+(0,2)+2(1,1)}$ $+[2,0,2]_{(\frac12,\frac12)} + [2,1,0]_{2(\frac12,\frac12)+(\frac12,\frac32)} + [2,2,0]_{(0,0)} + [3,0,1]_{(0,1)} + [1,0,1]_{4(\frac12,\frac12)+2(\frac12,\frac32)+2(\frac32,\frac12)+(\frac32,\frac32)}$
$\frac{11}{2}$	$[0,0,1]_{2(\frac12,0)+3(\frac12,1)+(\frac32,0)+2(\frac32,1)+(\frac32,2)} + [0,0,3]_{(0,\frac12)+(1,\frac12)} + [0,1,1]_{3(0,\frac12)+(0,\frac32)+4(1,\frac12)+2(1,\frac32)+(2,\frac12)}$ $+[0,1,3]_{(\frac12,0)} + [0,2,1]_{2(\frac12,0)+2(\frac12,1)+(\frac32,0)} + [0,3,1]_{(0,\frac12)} + [1,0,0]_{2(0,\frac12)+(0,\frac32)+3(1,\frac12)+2(1,\frac32)+(2,\frac32)}$ $+[1,0,2]_{2(\frac12,0)+2(\frac12,1)+(\frac32,0)+(\frac32,1)} + [1,1,0]_{3(\frac12,0)+4(\frac12,1)+(\frac12,2)+(\frac32,0)+2(\frac32,1)} + [1,1,2]_{(0,\frac12)+(1,\frac12)}$ $+[1,2,0]_{2(0,\frac12)+(0,\frac32)+2(1,\frac12)} + [1,3,0]_{(\frac12,0)} + [2,0,1]_{2(0,\frac12)+(0,\frac32)+2(1,\frac12)+(1,\frac32)} + [2,1,1]_{(\frac12,0)+(\frac12,1)}$ $+[3,0,0]_{(\frac12,0)+(\frac12,1)} + [3,1,0]_{(0,\frac12)}$
6	$[0,0,0]_{3(0,0)+3(1,1)+(2,2)} + [0,0,2]_{3(\frac12,\frac12)+(\frac12,\frac32)+(\frac32,\frac12)+(\frac32,\frac32)} + [0,1,0]_{4(\frac12,\frac12)+2(\frac12,\frac32)+2(\frac32,\frac12)+2(\frac32,\frac32)}$ $+[0,1,2]_{(0,0)+2(0,1)+2(1,0)+(1,1)} + [0,2,0]_{3(0,0)+(0,1)+(0,2)+(1,0)+3(1,1)+(2,0)} + [0,2,2]_{(\frac12,\frac12)}$ $+[0,3,0]_{2(\frac12,\frac12)} + [0,4,0]_{(0,0)} + [1,0,1]_{(0,0)+3(0,1)+3(1,0)+4(1,1)+(1,2)+(2,1)} + [1,0,3]_{(\frac12,\frac12)} + [0,0,4]_{(0,0)}$ $+[1,1,1]_{4(\frac12,\frac12)+2(\frac12,\frac32)+2(\frac32,\frac12)} + [1,2,1]_{(0,0)+(0,1)+(1,0)} + [2,0,0]_{3(\frac12,\frac12)+(\frac12,\frac32)+(\frac32,\frac12)+(\frac32,\frac32)}$ $+[2,0,2]_{(0,0)+(1,1)} + [2,1,0]_{(0,0)+2(0,1)+2(1,0)+(1,1)} + [2,2,0]_{(\frac12,\frac12)} + [3,0,1]_{(\frac12,\frac12)} + [4,0,0]_{(0,0)}$
$\frac{13}{2}$	$[0,0,1]_{2(0,\frac12)+(0,\frac32)+3(1,\frac12)+2(1,\frac32)+(2,\frac32)} + [0,0,3]_{(\frac12,0)+(\frac12,1)} + [0,1,1]_{3(\frac12,0)+4(\frac12,1)+(\frac12,2)+(\frac32,0)+2(\frac32,1)}$ $+[0,1,3]_{(0,\frac12)} + [0,2,1]_{2(0,\frac12)+(0,\frac32)+2(1,\frac12)} + [0,3,1]_{(\frac12,0)} + [1,0,0]_{2(\frac12,0)+3(\frac12,1)+(\frac32,0)+2(\frac32,1)+(\frac32,2)}$ $+[1,0,2]_{2(0,\frac12)+(0,\frac32)+2(1,\frac12)+(1,\frac32)} + [1,1,0]_{3(0,\frac12)+(0,\frac32)+4(1,\frac12)+2(1,\frac32)+(2,\frac12)} + [1,1,2]_{(\frac12,0)+(\frac12,1)}$ $+[1,2,0]_{2(\frac12,0)+2(\frac12,1)+(\frac32,0)} + [1,3,0]_{(0,\frac12)} + [2,0,1]_{2(\frac12,0)+2(\frac12,1)+(\frac32,0)+(\frac32,1)} + [2,1,1]_{(0,\frac12)+(1,\frac12)}$ $+[3,0,0]_{(0,\frac12)+(1,\frac12)} + [3,1,0]_{(\frac12,0)}$
7	$[0,0,0]_{(\frac12,\frac12)+(\frac12,\frac32)+(\frac32,\frac12)+(\frac32,\frac32)} + [0,0,2]_{2(0,0)+(0,1)+(0,2)+2(1,1)} + [0,1,0]_{3(0,1)+3(1,0)+2(1,1)+(1,2)+(2,1)}$ $+[0,1,2]_{2(\frac12,\frac12)+(\frac12,\frac32)} + [0,2,0]_{3(\frac12,\frac12)+(\frac12,\frac32)+(\frac32,\frac12)} + [0,2,2]_{(0,0)} + [0,3,0]_{(0,1)+(1,0)}$ $+[1,0,3]_{(0,1)} + [1,1,1]_{2(0,0)+2(0,1)+2(1,0)+2(1,1)} + [1,2,1]_{(\frac12,\frac12)} + [2,0,0]_{2(0,0)+(1,0)+2(1,1)+(2,0)}$ $+[2,0,2]_{(\frac12,\frac12)} + [2,1,0]_{2(\frac12,\frac12)+(\frac32,\frac12)} + [2,2,0]_{(0,0)} + [3,0,1]_{(1,0)} + [1,0,1]_{4(\frac12,\frac12)+2(\frac12,\frac32)+2(\frac32,\frac12)+(\frac32,\frac32)}$
$\frac{15}{2}$	$[0,0,1]_{(\frac12,0)+2(\frac12,1)+(\frac12,2)+(\frac32,0)+(\frac32,1)} + [0,0,3]_{(0,\frac32)} + [0,1,1]_{3(0,\frac12)+(0,\frac32)+2(1,\frac12)+(1,\frac32)} + [0,2,1]_{(\frac12,0)+(\frac12,1)}$ $+[1,0,0]_{(0,\frac12)+(0,\frac32)+2(1,\frac12)+(1,\frac32)+(2,\frac12)} + [1,0,2]_{(\frac12,0)+2(\frac12,1)} + [1,1,0]_{3(\frac12,0)+2(\frac12,1)+(\frac32,0)+(\frac32,1)}$ $+[1,1,2]_{(0,\frac12)} + [1,2,0]_{(0,\frac12)+(1,\frac12)} + [2,0,1]_{(0,\frac12)+2(1,\frac12)} + [2,1,1]_{(\frac12,0)} + [3,0,0]_{(\frac32,0)}$
8	$[0,0,0]_{(0,0)+(0,2)+(1,1)+(2,0)} + [0,0,2]_{(\frac12,\frac12)+(\frac12,\frac32)} + [0,1,0]_{2(\frac12,\frac12)+(\frac12,\frac32)+(\frac32,\frac12)} + [2,0,2]_{(0,0)} + [2,1,0]_{(1,0)}$ $+[0,1,2]_{(0,1)} + [0,2,0]_{2(0,0)+(1,1)} + [1,0,1]_{(0,0)+2(0,1)+2(1,0)+(1,1)} + [1,1,1]_{2(\frac12,\frac12)} + [2,0,0]_{(\frac12,\frac12)+(\frac32,\frac12)}$
$\frac{17}{2}$	$[0,0,1]_{(0,\frac12)+(0,\frac32)+(1,\frac12)} + [0,1,1]_{(\frac12,0)+(\frac12,1)} + [1,0,0]_{(\frac12,0)+(\frac12,1)+(\frac32,0)} + [1,0,2]_{(0,\frac12)}$ $+[1,1,0]_{(0,\frac12)+(1,\frac12)} + [2,0,1]_{(\frac12,0)}$
9	$[0,0,0]_{(\frac12,\frac12)} + [0,0,2]_{(0,0)} + [0,1,0]_{(0,1)+(1,0)} + [1,0,1]_{(\frac12,\frac12)} + [2,0,0]_{(0,0)}$
$\frac{19}{2}$	$[0,0,1]_{(\frac12,0)} + [1,0,0]_{(0,\frac12)}$
10	$[0,0,0]_{(0,0)}$

Table 1: Long Konishi multiplet $\mathcal{A}^2_{[0,0,0](0,0)}$.

saturate unitary bounds where long multiplets (3.8) become reducible and split into several semishort or BPS multiplets [28]. We give a brief review on this multiplet shortening in appendix C where protected $\frac{1}{2}$-BPS multiplets corresponding to the 'massless' supergravity states are also discussed.

Returning to the spectrum of (2.6), we note that at level $\ell = 1$, the SO(5) $\times$ SO(4) content of T_1 exactly coincides with the Konishi multiplet (3.9) upon breaking SO(6) down to SO(5). I.e. we are precisely in the situation discussed in (3.6). Hence, we can immediately write down the entire KK tower at $\ell = 1$ organized in supermultiplets (3.8) as

$$\mathcal{H}_1 = \sum_{n=0}^{\infty} [0,n,0]^n_{(0,0)} \times \hat{T}_1^{(2)} = \sum_{n=0}^{\infty} \mathcal{A}^{2+n}_{[0,n,0](0,0)} \,. \tag{3.10}$$

Unitarity (C.6) requires that $\Delta \geq 2+n$. The assignment $\Delta_0 = 2+n$ seems *ad hoc* at this level. We shall justify this choice later on. For string excitations at higher levels, the situation is more involved. Recall that for $\ell > 1$ the flat space spectrum was given in (2.6) as a product of T_1 with the SO(9) representation $(\mathrm{vac}_\ell)^2$. In order to obtain the KK tower in closed form (3.4), using the above result for T_1, it remains to lift $(\mathrm{vac}_\ell)^2$ to SU(4) $\times$ SO(4). At level $\ell = 2$, this may be achieved via

$$\mathbf{9} \to [0,1,0]^1_{(0,0)} + [0,0,0]^1_{(\frac{1}{2},\frac{1}{2})} - [0,0,0]^2_{(0,0)} \,. \tag{3.11}$$

This requires some comments. Note first that using this formal lift, the entire KK tower at level $\ell = 2$ may be written as

$$\begin{aligned}\mathcal{H}_2 = \sum_{n=0}^{\infty} [0,n,0]^n_{(0,0)} \times \hat{T}_1^{(2)} \times \Big(& [000]^2_{2(0,0)+(0,1)+(1,0)+(1,1)} + [000]^2_{(1,1)} + [020]^2_{(0,0)} + \\ & + [101]^2_{(0,0)} + 2\cdot[010]^2_{(\frac{1}{2},\frac{1}{2})} - 2\cdot[000]^3_{(\frac{1}{2},\frac{1}{2})} - \\ & - 2\cdot[010]^3_{(0,0)} + [000]^4_{(00)} \Big) \,. \end{aligned} \tag{3.12}$$

It may now be verified that the formal negative multiplicities in the last factor of this expression do not lead in fact to any states of negative multiplicity when multiplied with the infinite sum. E.g. the negative states [010] at $n = 0$, $\Delta = 3$ are precisely cancelled by the corresponding states at $n = 1$, etc. Note that this cancellation uniquely fixes the relative values of Δ for the HWS in the lift (3.11). Equation (3.12) thus yields a perfectly sensible result for the KK tower at level $\ell = 2$.

Higher levels allow similar decompositions

$$\mathcal{H}_\ell = \sum_{n=0}^{\infty} [0,n,0]^n_{(0,0)} \times \hat{T}_1^{(2)} \times (\widehat{\mathrm{vac}}_\ell)^2 \,, \tag{3.13}$$

where $\widehat{\mathrm{vac}}_\ell$ denotes the required lift of the flat space spectrum (2.6) to SU(4) $\times$ SO(4). Although increasing with the level, ambiguities in the lift of SO(5) to SO(6) affect only the conformal dimensions: the SO(6) $\times$ SO(4) decomposition of the KK towers is uniquely fixed following our procedure.

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Of particular interest in our subsequent discussions will be the sector of string states originating from the completely symmetric SO(9) representations $[\ell-1,0,0,0]$ appearing in vac_ℓ. Adopting the vector lift formula (3.11) for these tensors one finds:

$$[\ell-1,0,0,0] \to \sum_{t=0}^{\ell-1} [0,\ell-t-1,0]^{\ell}_{(\frac{t}{2},\frac{t}{2})} - \cdots \,, \tag{3.14}$$

with dots standing for higher energy states. Plugging this in (3.13) one arrives at a remarkably simple formula for the scaling dimensions:[8]

$$\Delta_0 = 2\ell + n \,, \tag{3.15}$$

for the corresponding HWS in the superstring spectrum. In the rest of the paper we will focus on this particularly interesting sector of the string spectrum and test the validity of (3.15). In particular the two extreme cases $s = t + \bar{t} = 2(\ell-1)$ and $t = \bar{t} = 0$ will be relevant for our discussions of HS currents and would-be $\frac{1}{4}$-BPS states respectively. We stress that (3.15) is expected to hold only for string states coming from (3.14), while the remaining states in $\widehat{\mathrm{vac}}_\ell$, the first appearing at $\ell = 3$, should be dealt with separately.

The commensurability of the contribution to (3.15) from the string level ℓ with the one from the KK harmonic n is consistent with our expectation that $R \approx \alpha'$ at $\lambda = 0$. Semiclassical formulae in inverse powers of $\sqrt{\lambda}$ valid for large s (often denoted by S) and/or n (often denoted by J) have been obtained for string solitons in $\mathrm{AdS}_5 \times S^5$ by worldsheet methods [21] that may be taken as a hint in extending (3.15) beyond the HS symmetric point. Clearly (3.15) is expected to only hold for the states appearing in (3.14). At higher level, $\ell \geq 3$, other kinds of states are present. In particular, it is well known that Δ grows as $\sqrt{\ell}$ for states maximizing it [56].

Finally, let us turn our attention on $\mathcal{H}_0$ that collects the KK descendants of massless type-IIB supergravity. States belonging to $\mathcal{H}_0$ organize in $\frac{1}{2}$-BPS multiplets $\mathcal{BB}^{\frac{1}{2},\frac{1}{2}}_{[0,n,0](0,0)}$ of $SU(2,2|4)$ (which we explicitly give in table 2 below)

$$\mathcal{H}_0 = \sum_{n=2}^{\infty} \mathcal{BB}^{\frac{1}{2},\frac{1}{2}}_{[0,n,0](0,0)} \,. \tag{3.16}$$

The omitted $n = 1$ and $n = 0$ multiplets correspond to the non-propagating singleton multiplet and AdS vacuum respectively. Supergravity KK recurrences have been extensively studied in the past starting with [47]. BPS multiplets are built by acting with half the supersymmetries on a chiral primary with conformal dimension Δ_0 saturating the unitary bound $\Delta_0[0,n,0]_{(0,0)} = n$. According to holography they are associated to SYM multiplets starting with the chiral primaries $\mathrm{Tr}\,\phi^{(i_1}\dots\phi^{i_n)}$. In particular, the $n = 2$ ground floor in the KK tower accounts for the 2^8 degrees of freedom of type-II supergravity as required.

[8] In global coordinates corresponding to SYM theory on $R^+ \times S^3$ where radial quantization exposes the state-operator correspondence, Δ can be identified with the 'energy' i.e. with the eigenvalue of the generator of time translation $H = P_0 + K_0$.

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Δ	
n	$[0,n,0]_{(0,0)}$
$n+\frac{1}{2}$	$[1,n-1,0]_{(0,\frac{1}{2})}+[0,n-1,1]_{(\frac{1}{2},0)}$
$n+1$	$[0,n-2,2]_{(0,0)}+[2,n-2,0]_{(0,0)}+[0,n-1,0]_{(0,1)}+[0,n-1,0]_{(1,0)}+[1,n-2,1]_{(\frac{1}{2},\frac{1}{2})}$
$n+\frac{3}{2}$	$[1,n-2,0]_{(0,\frac{1}{2})}+[1,n-3,2]_{(0,\frac{1}{2})}+[0,n-2,1]_{(\frac{1}{2},0)}+[2,n-3,1]_{(\frac{1}{2},0)}+$ $+[0,n-2,1]_{(\frac{1}{2},1)}+[1,n-2,0]_{(1,\frac{1}{2})}$
$n+2$	$2\cdot[0,n-2,0]_{(0,0)}+[2,n-4,2]_{(0,0)}+[0,n-3,2]_{(0,1)}+[2,n-3,0]_{(1,0)}+$ $+2\cdot[1,n-3,1]_{(\frac{1}{2},\frac{1}{2})}+[0,n-2,0]_{(1,1)}$
$n+\frac{5}{2}$	$[1,n-3,0]_{(0,\frac{1}{2})}+[1,n-4,2]_{(0,\frac{1}{2})}+[0,n-3,1]_{(\frac{1}{2},0)}+[2,n-4,1]_{(\frac{1}{2},0)}+$ $+[0,n-3,1]_{(\frac{1}{2},1)}+[1,n-3,0]_{(1,\frac{1}{2})}$
$n+3$	$[0,n-4,2]_{(0,0)}+[2,n-4,0]_{(0,0)}+[0,n-3,0]_{(0,1)}+[0,n-3,0]_{(1,0)}+[1,n-4,1]_{(\frac{1}{2},\frac{1}{2})}$
$n+\frac{7}{2}$	$[1,n-4,0]_{(0,\frac{1}{2})}+[0,n-4,1]_{(\frac{1}{2},0)}$
$n+4$	$[0,n-4,0]_{(0,0)}$

Table 2: BPS multiplet $\mathcal{BB}^{\frac{1}{2},\frac{1}{2}}_{[0,n,0](0,0)}$ with $n\geq 2$. Negative weights are understood to be omitted. The total number of states is $d(n)=2^8\cdot\dim[0,n-2,0]$.

3.2 Exploiting higher spin symmetry on AdS_5

As we mentioned in the previous section supersymmetry determines the spectrum of conformal dimensions (and therefore of "masses" M^2_Φ's) inside a supermultiplet once Δ is known for the HWS or for any other component. This reduces the task of computing M^2_Φ to fix Δ for a single component inside each supermultiplet. In this section we exploit the higher spin symmetries of $\mathcal{N}=4$ SYM at $\lambda=0$ in order to justify the bare conformal dimensions Δ_0 assigned in the previous section to KK descendants of higher string excitations on $\mathrm{AdS}_5\times S^5$. The new symmetries manifest themselves in the shortening of long multiplets saturating unitary bounds and containing particles that become massless in the free SYM limit.

Following [45] we define the AdS_5 mass of a given field belonging to the SO(4,2) representation[9] $D(\Delta,j,\bar{\jmath})$ by

$$m^2=C_2\left[\mathrm{SO}(4,2)\right]-m_0^2=\Delta(\Delta-4)-\Delta_{\rm min}(\Delta_{\rm min}-4)\,, \tag{3.17}$$

with

$$C_2\left[\mathrm{SO}(4,2)\right]=\Delta(\Delta-4)+2j(j+1)+2\bar{\jmath}(\bar{\jmath}+1)\,, \tag{3.18}$$

and m_0^2 a shift chosen in such a way that massless representations on AdS saturate one of the following bounds:[10]

$$\begin{aligned}
\Delta &\geq \Delta_{\rm min}\equiv 2+j+\bar{\jmath} && j,\bar{\jmath}\neq 0\,,\\
\Delta &\geq \Delta_{\rm min}\equiv 1+j && j\neq 0\,,\quad \bar{\jmath}=0\,,\\
\Delta &\geq \Delta_{\rm min}\equiv 1+\bar{\jmath} && j=0\,,\quad \bar{\jmath}\neq 0\,,\\
\Delta &\geq \Delta_{\rm min}\equiv 0 && j=\bar{\jmath}=0\,.
\end{aligned} \tag{3.19}$$

[9] SO(4,2) 'Dynkin labels' $[j,\Delta,\bar{\jmath}]$ are related to SO(6) Dynkin labels $[a,b,c]$ by $\Delta=b$, $j=\frac{1}{2}(a+c)$, $\bar{\jmath}=\frac{1}{2}(a-c)$.

[10] Obviously $\Delta_{\rm min}=1$ for scalar fields different from the identity operator.

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type	SO(4)	$\Delta_{\min}$	m^2
scalar	(0,0)	0	$\Delta(\Delta-4)$
vector	$(\frac{1}{2},\frac{1}{2})$	3	$(\Delta-1)(\Delta-3)$
Ant. tensor	$(1,0)$, $(0,1)$	2	$(\Delta-2)^2$
metric	$(1,1)$	3	$\Delta(\Delta-4)$
tensors	$(\frac{s}{2},\frac{s}{2})$	$2+s$	$(\Delta-2-s)(\Delta-2+s)$
	$(s,0)$, $(0,s)$	$1+s$	$(\Delta-1-s)(\Delta-3+s)$

Table 3: m^2 in AdS_5. $s \neq 0$ is understood.

We list in table 3 some relevant cases. In particular the SO(6) gauge vectors with $\Delta = 3$ and the graviton with $\Delta = 4$ account for the massless modes of $\mathcal{N} = 8$ supergravity i.e. $\ell = 0$.

We are interested in the free SYM limit that exhibits an enhanced $hs(2,2|4,\mathbb{R})$ higher spin symmetry. According to holography this should correspond to higher spin particles in AdS_5 becoming massless as $\lambda \to 0$. Unlike the SO(6) gauge vectors and the metric tensor of supergravity, the gauge particles realizing this higher spin symmetry sit in semishort (rather than $\frac{1}{2}$-BPS) multiplets and can become massive by 'eating' Goldstone particles when the dilaton takes a VEV. The boundary conformal field theory counterpart of this quantum effect is the emergence of anomalous dimensions[11] showing up in logarithmic behaviors at short distances [23, 24]. The higher spin currents are indeed conserved only in the strict $\lambda = 0$ limit where the SO(4,2) unitary bound $\Delta_0 = 2 + j + \bar{\jmath}$ be saturated. It is easy to identify these currents in the string spectrum (3.13). They correspond to multiplets built out of the highest spin components $(\mathrm{vac}_\ell)^2 = [000]_{(\ell-1,\ell-1)} + \cdots$, described in (3.14). Masslessness in AdS_5 requires that the bound $\Delta_0 = 2\ell = 2 + s$ is saturated by the HWS, confirming the *ad hoc* assignment (3.15). The corresponding long multiplets decompose into a sum of BPS/semishort ones according to (C.8)

$$\mathcal{A}^{2\ell}_{[000](\ell-1,\ell-1)^*} \approx \mathcal{CC}^{1,1}_{[000](\ell-1,\ell-1)^*} + \cdots . \tag{3.20}$$

Higher spin conserved currents sit in $\mathcal{CC}^{1,1}_{[000](\ell-1,\ell-1)^*}$. Here and below we denote massless representations by $(j,\bar{\jmath})^*$ defined after the subtraction of the gauge degrees of freedom $(j-\frac{1}{2},\bar{\jmath}-\frac{1}{2})$. The details can be found in [28] or in appendix C. The content of gauge multiplets is displayed in table 4 in the appendix. Taking into account that all fields in this multiplet satisfy massless field equations, the total dimension of this multiplet is given by $256\,(2\ell+3)$. The spectrum agrees with that of the massless higher spin theory studied in [8] and realized in terms of a single massless multiplet of the higher spin algebra $hs(2,2|4)$. Linearized field equations for these fields in AdS_5 have been worked it out in [8]. The remaining multiplets in the decomposition include the Goldstone particles needed for the Higgsing of the higher spin gauge symmetry at $\lambda \neq 0$.

KK descendants with $n > 0$ of conserved currents sitting inside $\mathcal{A}^{\Delta_0}_{[0n0](\ell-1,\ell-1)}$ do no longer satisfy the $SO(2,4)$ unitarity bound but they rather realize the $SU(2,2|4)$

[11] Semishort [48] and 1/4 BPS multi-trace operators [49] dual to multi-particle states at threshold remain short if there are no 'state/operators' to pair with.

Δ	
2ℓ	$[0,0,0]_{(\ell-1,\ell-1)}$
$2\ell+\frac{1}{2}$	$[0,0,1]_{(\ell-1,\ell-\frac{1}{2})}+[1,0,0]_{(\ell-\frac{1}{2},\ell-1)}$
$2\ell+1$	$[0,1,0]_{(\ell-1,\ell)}+[0,1,0]_{(\ell,\ell-1)}+[0,0,0]_{(\ell-\frac{1}{2},\ell-\frac{1}{2})}+[1,0,1]_{(\ell-\frac{1}{2},\ell-\frac{1}{2})}$
$2\ell+\frac{3}{2}$	$[1,0,0]_{(\ell-1,\ell+\frac{1}{2})+(\ell,\ell-\frac{1}{2})}+[0,0,1]_{(\ell+\frac{1}{2},\ell-1)+(\ell-\frac{1}{2},\ell)}+[1,1,0]_{(\ell-\frac{1}{2},\ell)}+[0,1,1]_{(\ell,\ell-\frac{1}{2})}$
$2\ell+2$	$[0,0,0]_{(\ell-1,\ell+1)+(\ell,\ell)+(\ell+1,\ell-1)}+[0,1,0]_{(\ell-\frac{1}{2},\ell+\frac{1}{2})+(\ell+\frac{1}{2},\ell-\frac{1}{2})}+[1,0,1]_{(\ell,\ell)}+[0,2,0]_{(\ell,\ell)}+$ $+[2,0,0]_{(\ell-\frac{1}{2},\ell+\frac{1}{2})}+[0,0,2]_{(\ell+\frac{1}{2},\ell-\frac{1}{2})}$
$2\ell+\frac{5}{2}$	$[1,0,0]_{(\ell-\frac{1}{2},\ell+1)+(\ell+\frac{1}{2},\ell)}+[0,0,1]_{(\ell+1,\ell-\frac{1}{2})+(\ell,\ell+\frac{1}{2})}+[1,1,0]_{(\ell,\ell+\frac{1}{2})}+[0,1,1]_{(\ell+\frac{1}{2},\ell)}$
$2\ell+3$	$[0,1,0]_{(\ell,\ell+1)}+[0,1,0]_{(\ell+1,\ell)}+[0,0,0]_{(\ell+\frac{1}{2},\ell+\frac{1}{2})}+[1,0,1]_{(\ell+\frac{1}{2},\ell+\frac{1}{2})}$
$2\ell+\frac{7}{2}$	$[0,0,1]_{(\ell+\frac{1}{2},\ell+1)}+[1,0,0]_{(\ell+1,\ell+\frac{1}{2})}$
$2\ell+4$	$[0,0,0]_{(\ell+1,\ell+1)}$

Table 4: Massless multiplet $\mathcal{CC}^{1,1}_{[0,0,0](\ell-1,\ell-1)^*}$. The number of physical states is $d(\ell)=2^8\cdot(4\ell+1)$.

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bound (C.6):

$$\Delta_0 = 2\ell + n\,.$$

This condition generalizes to $\mathrm{AdS}_5\times S^5$ the massless bound in AdS_5. Long multiplets starting with HWS saturating this bound decompose again in terms of semishort multiplets according to (C.8). The pattern of shortenings in (C.8) shows similarities but at the same time some differences for the Konishi multiplet, its HS 'relatives' and their KK excitations. The celebrated $\mathcal{N}=4$ Konishi anomaly [22] translates into a violation of a linear type shortening condition [25]. The KK excitations of the Konishi multiplet are 'violated' by a generalized Konishi anomaly that can be easily deduced from the results of [50] by observing that $\mathcal{N}=4$ SYM is a very special $\mathcal{N}=1$ SYM theory with three chiral multiplets in the adjoint. The role of this $\mathcal{N}=4$ generalized Konishi anomaly in the mixing of BMN operators [7] with two-impurities will be analyzed in [51]. Finally, violation of HS currents in $\mathcal{N}=4$ SYM requires Goldstone multiplets with fermionic HWS that have so far been largely unexplored.

We would also like to stress once more that the full superstring spectrum decomposes into infinitely many (generically 'massive') HS multiplets. In particular the first Regge trajectory consists of a single 'massless' HS gauge fields dual, in the holographic sense, to twist-two bilinear currents together with their superpartners. The first KK recurrence and (part of) the Goldstones belong to two different and massive HS multiplets dual to twist-three trilinear operators. At the next level one finds, in addition to the second KK recurrence and the remaining Goldstones new massive HS multiplets all of which are dual to twist-four quadrilinear operators. Proceeding to higher levels is straightforward in principle but requires a better understanding of massive HS multiplets.

4. The spectrum of free $\mathcal{N}=4$ SYM theory

In this section we derive the spectrum of single trace operators in $\mathcal{N}=4$ SYM theory with group $\mathrm{SU}(N)$ at large N. Single trace operators are associated to 'words' built with

Field	SU(4)	$(j,\bar{\jmath})$	Δ
ϕ^i	[010]	(0,0)	1
$\partial_{\alpha\dot{\alpha}}$	[000]	$(\frac{1}{2},\frac{1}{2})$	1
λ^A_α	[100]	$(\frac{1}{2},0)$	$\frac{3}{2}$
$\lambda_{\dot{\alpha}A}$	[001]	$(0,\frac{1}{2})$	$\frac{3}{2}$
$F_{\alpha\beta}$	[000]	$(1,0)$	2
$F_{\dot{\alpha}\dot{\beta}}$	[000]	$(0,1)$	2

Table 5: $\mathcal{N}=4$ SYM multiplet.

the 'alphabet' consisting of the 'letters'[12] [42, 52] ϕ^i, with $i=1,\dots,6$, λ^A_α and $\lambda_{\dot{\alpha}A}$, with $A=1,\dots,4$ and $\alpha,\dot{\alpha}=1,2$, $F_{\mu\nu}$, with $\mu=0,\dots,3$, and derivatives thereof. As usual $V_{\alpha\dot{\alpha}}=V_\mu\sigma^\mu_{\alpha\dot{\alpha}}$, $A_{\alpha\beta}=\frac{1}{2}A_{\mu\nu}\sigma^{\mu\nu}_{\alpha\beta}$ and similarly for $A_{\dot{\alpha}\dot{\beta}}$.

The quantum numbers of a given operator at $\lambda=0$ can be read off from those of the building letters collected in table 5.

4.1 Enumerating SYM operators: Polya(kov) theory

Enumerating SYM states can look at first sight a rather tedious exercise but it can be conveniently handled by resorting to Polya theory [41]. This idea was applied by A. Polyakov in [42] to the counting of gauge invariant operators made out of bosonic 'letters'. Here we extend his results and enumerate, rather than simply count, gauge invariant words including fermionic letters, see also [46]. For future reference, we work out the counting from a more general perspective than what we really need. This section is written in a self-consistent form and can be read independently of the rest of the paper.

Lets start by briefly reviewing the basics of Polya theory. Consider a set of words $A,B,\dots$ made out of n letters chosen within the alphabet $\{a_i\}$ with $i=1,\dots p$. Let G be a group action defining the equivalence relation $A\sim B$ for $A=gB$ with g an element of $G\subset S_n$. Elements $g\in S_n$ can be divided into conjugacy classes $[g]=(1)^{b_1}\dots(n)^{b_n}$, according to the numbers $\{b_k(g)\}$ of cycles of length k. Polya theorem states that the set of inequivalent words are generated by the formula:

$$P_G(\{a_i\})\equiv\frac{1}{|G|}\sum_{g\in G}\prod_{k=1}^{n}(a_1^k+a_2^k+\cdots+a_p^k)^{b_k(g)}\,. \tag{4.1}$$

In particular, for $G=Z_n$, the cyclic permutation subgroup of S_n, the elements $g\in G$ belong to one of the conjugacy classes $[g]=(d)^{\frac{n}{d}}$ for each divisor d of n. The number of elements in a given conjugacy class labeled by d is given by Euler's totient function $\varphi(d)$, equal to the number of numbers relatively prime to n (to be pedantic) smaller than n. For $n=1$ one defines $\varphi(1)=1$. Computing P_G for $G=Z_n$ one finds:

$$P_n(\{a_i\})\equiv\frac{1}{n}\sum_{d|n}\varphi(d)(a_1^d+a_2^d+\cdots+a_p^d)^{\frac{n}{d}}\,. \tag{4.2}$$

[12] We adhere to the nomenclature introduced by A. Polyakov although the terms 'necklaces' (for 'words') and 'beads' (for 'letters') sound more romantic and to some extent ('cyclicity') more appropriate.

The number of inequivalent words can be read off from (4.1) by simply letting $a_i \to 1$. For instance, the possible choices of 'necklaces' with six 'beads' of two different 'colors' a and b, are given by

$$\begin{aligned} P_6(a,b) &= \frac{1}{6}\left[(a+b)^6 + (a^2+b^2)^3 + 2(a^3+b^3)^3 + 2(a^6+b^6)\right] \\ &= a^6 + a^5 b + 3a^4 b^2 + 4a^3 b^3 + 3a^2 b^4 + ab^5 + b^6\,, \end{aligned}$$

and the number of different necklaces is $P_6(a=b=1) = 14$.

Let now come to our main task: To count the number of gauge invariant words (single trace operators) in four-dimensional SU(N) YM theories with fields in the adjoint representation.[13] As mentioned at the beginning we will endeavor to be rather general and consider words made out of the (on-shell) letters:

$$\begin{aligned} &\partial^s\phi \in \mathbf{n_s}{}^{(s+1)}_{(\frac{s}{2},\frac{s}{2})}\,, &&\partial^s F \in \mathbf{n_v}{}^{(s+2)}_{(\frac{s+2}{2},\frac{s}{2})} + \mathbf{n_v}{}^{(s+2)}_{(\frac{s}{2},\frac{s+2}{2})}\,, \\ &\partial^s\lambda \in \mathbf{n_f}{}^{s+\frac{3}{2}}_{(\frac{s+1}{2},\frac{s}{2})}\,, &&\partial^s\bar\lambda \in \mathbf{n_{\bar f}}{}^{s+\frac{3}{2}}_{(\frac{s}{2},\frac{s+1}{2})}\,, \end{aligned} \tag{4.3}$$

accounting for the elementary YM fields and derivatives thereof. On-shell conditions have been taken care of by restricting the action of the derivatives ∂^s to completely symmetric and traceless representation of the $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ Lorentz group. Finally $\mathbf{n_s}, \mathbf{n_f}, \mathbf{n_{\bar f}}, \mathbf{n_v}$ refers to the number of scalars, spinor left, spinor right and vector, respectively. In most cases these labels denote (reducible) representations of a global symmetry group.

In order to count words one has to take into account the cyclicity of the trace. This is exactly what the cycle index (4.2) does for us. The only subtleties are the arbitrary number of letters in a given word, the infinite number of letters in the alphabet and the statistics of the letters. It is easy to convince oneself that

$$\begin{aligned} \mathcal{Z}_{\mathrm{YM}}(q) &= \sum_{n,d|n} \frac{\varphi(d)}{n}\left[\mathcal{F}_s^{(d)}(q) + \mathcal{F}_v^{(d)}(q) - \mathcal{F}_f^{(d)}(q) - \mathcal{F}_{\bar f}^{(d)}(q)\right]^{\frac{n}{d}} \\ &= -\sum_d \frac{\varphi(d)}{d}\log[1 - \mathcal{F}_s^{(d)}(q) - \mathcal{F}_v^{(d)}(q) + \mathcal{F}_f^{(d)}(q) + \mathcal{F}_{\bar f}^{(d)}(q)]\,, \end{aligned} \tag{4.4}$$

does the job for U(N) at large N. For SU(N) one simply eliminates words with only one letter, i.e. $n \geq 2$. Finally the sign takes care of the right spin statistics. In particular, fermion states will appear at half-integer powers of q with an overall negative sign. Incidentally we notice that is exactly the way one derives Fermi statistics.

Following [42, 52] we define the partition functions of each species

$$\mathcal{F}_\Phi^{(d)}(q) = \sum_{s,I} q^{d(s+\Delta_\Phi)} (\partial^s\Phi_I)^d\,, \tag{4.5}$$

where I is a multi-index encompassing spin and 'flavour' in (4.3) while the parameter q traces the contribution of a given letter to the conformal dimension Δ.

[13] Other gauge groups and/or other representations for the matter field require a little more effort and will not be considered here.

Obviously replacing all $\partial^s \Phi_I$ by 1, one is simply counting the on-shell 'letters' of a given species. Explicitly for the letters in (4.3) and $D=4$ one finds:

$$\begin{aligned}
\mathcal{F}_s^{(d)}(q) &\to n_s \sum_{s=0}^{\infty}(s+1)^2 q^{d(s+1)} = n_s \frac{q^d(1+q^d)}{(1-q^q)^3}\,, \\
\mathcal{F}_f^{(d)}(q) &\to n_f \sum_{s=0}^{\infty}(s+1)(s+2) q^{d(s+\frac{3}{2})} = 2\,n_f \frac{q^{\frac{3d}{2}}}{(1-q^d)^3}\,, \\
\mathcal{F}_{\bar f}^{(d)}(q) &\to n_{\bar f} \sum_{s=0}^{\infty}(s+2)(s+1) q^{d(s+\frac{3}{2})} = 2\,n_{\bar f} \frac{q^{\frac{3d}{2}}}{(1-q^d)^3}\,, \\
\mathcal{F}_v^{(d)}(q) &\to 2\,n_v \sum_{s=0}^{\infty}(s+1)(s+3) q^{d(s+2)} = 2\,n_v \frac{q^{2d}(3-q^d)}{(1-q^d)^3}\,,
\end{aligned} \tag{4.6}$$

with $n_s, n_f, n_{\bar f}, n_v$ defined above.

It is now an easy algebraic exercise to produce explicit formulae for the partition functions of pure $\mathcal{N}$-supersymmetric free YM theories, for which

$$\begin{aligned}
&\mathcal{N}=4 \qquad n_s=6\,, \quad n_f=n_{\bar f}=4\,, \quad n_v=1\,, \\
&\mathcal{N}=2 \qquad n_s=2\,, \quad n_f=n_{\bar f}=2\,, \quad n_v=1\,, \\
&\mathcal{N}=1 \qquad n_s=0\,, \quad n_f=n_{\bar f}=1\,, \quad n_v=1\,.
\end{aligned} \tag{4.7}$$

Plugging into (4.4) and expanding in powers of q up to $\Delta=4$:

$$\begin{aligned}
\mathcal{Z}_{\mathcal{N}=4}(q) &= 21\,q^2 - 96\,q^{\frac{5}{2}} + 361\,q^3 - 1328\,q^{\frac{7}{2}} + 4601\,q^4 + \cdots\,, \\
\mathcal{Z}_{\mathcal{N}=2}(q) &= 3\,q^2 - 16\,q^{\frac{5}{2}} + 57\,q^3 - 184\,q^{\frac{7}{2}} + 551\,q^4 + \cdots\,, \\
\mathcal{Z}_{\mathcal{N}=1}(q) &= 6\,q^3 - 20\,q^{\frac{7}{2}} + 65\,q^4 + \cdots
\end{aligned} \tag{4.8}$$

In (4.8) we have subtracted unphysical modes $\mathcal{Z}_{\rm unphys} = (n_f n_{\bar f} - 1)q^3 + 4q^4 - 2(n_f + n_{\bar f})q^{\frac{7}{2}}$ but not total derivatives (conformal descendants).

4.2 Marginal and relevant SYM operators

Specializing to $\mathcal{N}=4$ SYM, reinstating the 'letters' in (4.4) and expanding up to $\Delta=4$ but keeping only integer powers of q we are left with the bosonic generating function:

$$\begin{aligned}
\mathcal{Z}_{\rm bos}(q) = {}& \frac{1}{2}(\phi_i\phi_j + \phi_i^2)\,q^2 + \\
&+\left[\frac{1}{3}(\phi_i\phi_j\phi_k + 2\phi_i^3) + \frac{1}{2}(\bar\lambda_{\dot\alpha A}\bar\lambda_{\dot\beta B} - \bar\lambda_{\dot\alpha A}^2 + \text{h.c.}) + \lambda_\alpha^A\bar\lambda_{\dot\alpha B} + F_{\mu\nu}\phi_i + \phi_i\partial_\mu\phi_j\right] q^3 + \\
&+\Big[\frac{1}{4}(\phi_{i_1}\phi_{i_2}\phi_{i_3}\phi_{i_4} + \phi_{i_1}^2\phi_{i_2}^2 + 2\phi_{i_1}^4) + (\lambda_\alpha^A + \bar\lambda_{\dot\alpha A})^2\phi_i + (\lambda_\alpha^A + \bar\lambda_{\dot\alpha A})\partial_\mu(\lambda_\beta^B + \bar\lambda_{\dot\beta B}) + \\
&\quad + \frac{1}{2}(F_{\mu\nu}F_{\sigma\rho} + F_{\mu\nu}^2) + F_{\mu\nu}\phi^i\phi^j + \phi_i\phi_j\partial_\mu\phi_k + \phi_i\partial_\mu F_{\mu\sigma} + \partial_\mu\phi_i F_{\mu\sigma} + \\
&\quad + \frac{1}{2}(\partial_\mu\phi_i\partial_\nu\phi_j + (\partial_\mu\phi_i)^2)) + \phi_i\partial_\mu\partial_\nu\phi_j\Big] q^4 + \cdots
\end{aligned} \tag{4.9}$$

Δ	$(j,\bar{\jmath})$	SU(4)	$\mathcal{O}$
2	(0,0)	[0,0,0]+[0,2,0]=**1**+**20**	$\mathrm{Tr}\,\phi^{((i_1}\phi^{i_2))}$
3	(0,0)	[0,1,0]+[0,3,0]=**6**+**50**	$\mathrm{Tr}\,\phi^{((i_1}\phi^{i_2}\phi^{i_3))}$
	(0,0)	[0,0,2]+[0,0,2]=$\mathbf{10}_s$+$\mathbf{10}_c$	$\mathrm{Tr}\,\phi^{[i_1}\phi^{i_2}\phi^{i_3]}$
	(0,0)	[2,0,0]+[0,0,2]=$\mathbf{10}_s$+$\mathbf{10}_c$	$\mathrm{Tr}\,\lambda^{(A}_\alpha\lambda^{B)\alpha}$ + h.c.
	(1,0)	[0,1,0]=**6**	$\mathrm{Tr}\,F_{\alpha\beta}\,\phi^i$
	(1,0)	[0,1,0]= **6**	$\mathrm{Tr}\,\lambda^{[A}_{(\alpha}\lambda^{B]}_{\beta)}$
	$(\frac{1}{2},\frac{1}{2})^*$	[1,0,1]=**15**	$\mathrm{Tr}\,\phi^{[i_1}\partial_\mu\phi^{i_2]}$
	$(\frac{1}{2},\frac{1}{2})^*$	[0,0,0]+[1,0,1]=**1**+**15**	$\mathrm{Tr}\,\lambda^A_\alpha\bar{\lambda}_{\dot{\beta}B}$
4	(0,0)	[0,0,0]+[0,2,0]+[0,4,0]= **1**+**20**+**105**	$\mathrm{Tr}\,\phi^{((i_1}\phi^{i_2}\phi^{i_3}\phi^{i_4))}$
	(0,0)	[0,0,0]+[0,2,0]+[2,0,2]= **1**+**20**+**84**	$\mathrm{Tr}\,\phi^{[i_1}\phi^{i_2]}\phi^{[i_3}\phi^{i_4]}$
	(0,0)	[1,0,1]+[0,1,2]+[2,1,0]= **15**+$\mathbf{45}_s$+$\mathbf{45}_c$	$\mathrm{Tr}\,\phi^{[i_1}\phi^{i_2}\phi^{((i_3]}\phi^{i_4))}$
	(0,0)	2([000]+[1,0,1]+[0,2,0]) =2(**1**+**15**+**20**)	$\mathrm{Tr}\,\lambda^{[A}_\alpha\lambda^{B]\alpha}\phi^i$ + h.c.
	(0,0)	2[1,0,1]+[0,1,2]+[2,1,0]=2·**15**+$\mathbf{45}_s$+$\mathbf{45}_c$	$\mathrm{Tr}\,\lambda^{(A}_\alpha\lambda^{B)\alpha}\phi^i$ + h.c.
	(0,0)	2[0,0,0]= 2·**1**	$\mathrm{Tr}\,F^2$, $\mathrm{Tr}\,F\widetilde{F}$
	(1,0)	[000]+[1,0,1]+[0,2,0]=**1**+**15**+**20**	$\mathrm{Tr}\,\lambda^{[A}_{(\alpha}\lambda^{B]}_{\beta)}\phi^i$
	(1,0)	[1,0,1]+[2,1,0] = **15**+$\mathbf{45}_s$	$\mathrm{Tr}\,\lambda^{(A}_{(\alpha}\lambda^{B)}_{\beta)}\phi^i$
	(1,0)	[0,0,0]+[1,0,1]+[0,2,0]= **1**+**15**+**20**	$\mathrm{Tr}\,F_{\alpha\beta}\,\phi^{i_1}\phi^{i_2}$
	$(\frac{1}{2},\frac{1}{2})$	2[1,1,1]+2[0,1,0]= 2·**6**+2·**64**	$\mathrm{Tr}\,\partial_\mu\phi^{((i_1}\phi^{[i_2))}\phi^{i_3]}$
	$(\frac{1}{2},\frac{1}{2})$	4[010]+2[0,0,2]+2[2,0,0]+2[1,1,1]=	$\mathrm{Tr}\,\lambda^A_\alpha\bar{\lambda}_{\dot{\beta}B}\phi^i$, $\mathrm{Tr}\,\lambda^A_\alpha\phi^i\bar{\lambda}_{\dot{\beta}B}$
		=4·**6**+2·$\mathbf{10}_s$+2·$\mathbf{10}_c$+ 2·**64**	
	$(\frac{3}{2},\frac{1}{2})^*$	[2,0,0] =$\mathbf{10}_s$	$\mathrm{Tr}\,\lambda^{(A}_{(\alpha}\partial_{\dot{\alpha}\beta}\lambda^{B)}_{\gamma)}$
	$(\frac{3}{2},\frac{1}{2})^*$	[0,1,0] =**6**	$\mathrm{Tr}\,\partial_{\dot{\alpha}(\gamma}F_{\alpha\beta)}\phi^i$
	(2,0)	[0,0,0]= **1**	$\mathrm{Tr}\,F_{(\alpha\beta}F_{\gamma\delta)}$
	$(1,1)^*$	[0,0,0]= **1**	$\mathrm{Tr}\,F_{\alpha\beta}F_{\dot{\alpha}\dot{\beta}}$
	$(1,1)^*$	[0,0,0]+[0,2,0]=**1**+**20**	$\mathrm{Tr}\,\partial_{(\mu}\phi^{((i_1}\,\partial_{\nu)}\,\phi^{i_2))}$
	$(1,1)^*$	[0,0,0]+[1,0,1]=**1**+**15**	$\mathrm{Tr}\,\lambda^A_{(\alpha}\partial_{\beta)(\dot{\beta}}\bar{\lambda}_{\dot{\alpha})B}$

Table 6: $\mathcal{N}=4$ SYM at $\lambda=0$. Brackets denote antisymmetrization. Parentheses denote complete symmetrization when traces cannot appear. Double parentheses denote complete symmetrization not excluding traces.

The sum over Lorentz and flavor indices is as before implicitly understood. As mentioned before (4.9) is valid for SU(N) at large N and $\lambda=0$. Single-letter words have been discarded.

Next we organize states in irreducible representations of SO(4) $\times$ SU(4). The SU(4) content of (4.9) can be read off by letting

$$\begin{aligned} \frac{1}{2}(\phi_i\phi_j+\phi_i^2) &\rightarrow \phi^{((i}\phi^{j))} \\ \frac{1}{3}(\phi_i\phi_j\phi_k+2\,\phi_i^3) &\rightarrow \phi^{((i_1}\phi^{i_2}\phi^{i_3))}+\phi^{[i_1}\phi^{i_2}\phi^{i_3]} \\ \frac{1}{2}(\lambda^A_\alpha\lambda^B_\beta-\lambda^{A\,2}_\alpha) &\rightarrow \lambda^{(A}_{[\alpha}\lambda^{B)}_{\beta]}+\lambda^{[A}_{(\alpha}\lambda^{B]}_{\beta)}\,, \qquad \text{and so on}\,, \end{aligned} \tag{4.10}$$

as always traces are understood. For instance, the multiplicities of states of the form $\phi^1\phi^1\phi^1$, $\phi^1\phi^1\phi^2$, $\phi^1\phi^2\phi^3$ in the left hand side of (4.10) are $1,1,2$ respectively determining the right hand side. This can be confirmed by counting the number of cyclic words of the corresponding type. In a similar way one can work out the other cases. The final output for single-trace operators with $\Delta_0\leq 4$ is given in table 6. In building table 6 we omit

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total derivatives. In addition we discard terms containing $\partial^2\ \phi^i, \not\partial\ \lambda^A, \not\partial\ \bar\lambda_A, \partial^\mu\ F_{\mu\nu}$, or $\partial^\mu\ \widetilde{F}_{\mu\nu}$, which vanish along the 'free' field equations or after imposing Bianchi identities. In particular, the equations of motion imply the conservation of all $s=1$ currents at $\Delta=3$ and $s=2$ tensors at $\Delta=4$. Special cases of this are the SO(6) current at $\Delta=3$ and the stress energy tensor at $\Delta=4$ realized in supergravity which are the only currents whose conservation survives quantum corrections.

5. String vs. gauge theory

5.1 Operators/states with $\Delta \leq 4$

Let us now identify the string excitations which are dual to the relevant and marginal operators found in the previous section. According to (3.15) string states with $\Delta \leq 4$ appear only in $\mathcal{H}_\ell$ with $\ell = 0,1,2$. Collecting (3.10), (3.12), (3.16) we find:

$$\begin{aligned}\mathcal{H}_{\Delta\leq 4} \in \sum_{n=2}^{4} \mathcal{BB}^{\frac{1}{2},\frac{1}{2}}_{[0n0](00)} + \sum_{n=0}^{2} \mathcal{A}^{2+n}_{[0n0](00)} + 2\mathcal{A}^{4}_{[000](00)} + \mathcal{A}^{4}_{[101](00)} + \mathcal{A}^{4}_{[020](00)} \\ + \mathcal{A}^{4}_{[000](10)} + \mathcal{A}^{4}_{[000](01)} + 2\mathcal{A}^{4}_{[010](\frac{1}{2},\frac{1}{2})} + \mathcal{A}^{4}_{[000](11)}\,. \end{aligned} \tag{5.1}$$

The first two terms represent the contributions of supergravity and Konishi together with their lowest KK recurrences. The remaining multiplets appears at $\ell = 2$. The genuinely long supermultiplets $\mathcal{A}^{4}_{[000](00)}$ have been studied in [26, 27, 38, 39]. Beside these, the remaining long multiplets in (5.1) saturate at $\lambda = 0$ the unitarity bounds (C.6) and split into a sum of BPS/semishort multiplets according to (C.8). The resulting spectrum of $\Delta \leq 4$ string states is collected in tables 7, 8. Comparing the multiplicities and quantum numbers in the string/sugra and SYM tables we find precise agreement! Establishing this one-to-one correspondence between the spectrum of relevant and marginal deformations of string and SYM theories provides strong support to the dimension formula (3.15). This remarkably neat formula remains on a conjectural basis and will require extensions in order to deal with fermionic and other HWS, not of the type (3.14), present at higher levels. However we find it a very appealing starting point for future analyses of the string spectrum on $\mathrm{AdS}_5 \times S^5$ especially at the point of minimal radius where the theory drastically simplify [53, 9, 10].

5.2 BPS and semishort multiplets

We would like to focus now on string/SYM states that belong to "nearly protected" long multiplets of the AdS supergroup, i.e. multiplets starting with a highest weight state $[k,p,q]_{(j,\bar\jmath)}$ saturating unitarity bounds at $\lambda = 0$. For simplicity we restrict ourselves to the series $k - q = j - \bar\jmath = 0$. A self-consistent description of the decomposition of such multiplets into BPS/semishort multiplets of SU(2,2|4) is presented in appendix C along the lines of [28]. Here we would like to count multiplets falling in this particularly interesting class.

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Multiplet	Δ	SU(4)	$(j,\bar{\jmath})$
$\mathcal{CC}^{1,1}_{[000](0,0)^*}$	2	[000]	$(0,0)$
	3	[010]	$(0,1)+(1,0)$
		[000] + [101]	$(\frac{1}{2},\frac{1}{2})^*$
	4	[000]	$(2,0)+(0,2)$
		[000] + [101] + [020]	$(1,1)^*$
		[002] + [010]	$(\frac{3}{2},\frac{1}{2})^*$
		[200] + [010]	$(\frac{1}{2},\frac{3}{2})^*$
$\mathcal{BC}^{\frac{1}{4},\frac{3}{4}}_{[200](0,0)}$	3	[200]	$(0,0)$
	4	[000] + [101] + [020]	$(0,0)$
		[210]	$(0,1)$
		[101]	$(1,0)+(0,1)$
		[111] + [200] + [010]	$(\frac{1}{2},\frac{1}{2})$
$\mathcal{CB}^{\frac{3}{4},\frac{1}{4}}_{[002](0,0)}$	3	[002]	$(0,0)$
	4	[000] + [101] + [020]	$(0,0)$
		[012]	$(1,0)$
		[101]	$(1,0)+(0,1)$
		[111] + [002] + [010]	$(\frac{1}{2},\frac{1}{2})$
$\mathcal{BB}^{\frac{1}{4},\frac{1}{4}}_{[202](0,0)}$	4	[202]	$(0,0)$
$\mathcal{CC}^{\frac{1}{2},\frac{1}{2}}_{[010](0,0)}$	3	[010]	$(0,0)$
	4	2·[101]	$(0,0)$
		[111] + 2·[010] + [200] + [002]	$(\frac{1}{2},\frac{1}{2})$
		[000] + [101] + [020]	$(1,0)+(0,1)$
$\mathcal{BC}^{\frac{1}{4},\frac{1}{2}}_{[210](0,0)}$	4	[210]	$(0,0)$
$\mathcal{CB}^{\frac{1}{2},\frac{1}{4}}_{[012](0,0)}$	4	[012]	$(0,0)$
$\mathcal{CC}^{\frac{1}{2},\frac{1}{2}}_{[020](0,0)}$	4	[020]	$(0,0)$
$\mathcal{CC}^{1,1}_{[000](1,1)^*}$	4	[000]	$(1,1)^*$
$\mathcal{CC}^{\frac{1}{2},\frac{1}{2}}_{[020](0,0)}$	4	[020]	$(0,0)$
$\mathcal{CC}^{\frac{1}{4},\frac{1}{4}}_{[101](0,0)}$	4	[101]	$(0,0)$
$\mathcal{CC}^{\frac{1}{2},\frac{1}{2}}_{[010](\frac{1}{2},\frac{1}{2})}$	4	[010]	$(\frac{1}{2},\frac{1}{2})$
$\mathcal{AC}^{1}_{[000](0,1)}$	4	[000]	$(0,1)$
$\mathcal{CA}^{1}_{[000](1,0)}$	4	[000]	$(1,0)$
$2\cdot\mathcal{A}^{4}_{[000](0,0)}$	4	[000]	$(0,0)$

Table 7: Bosonic modes with $\Delta \leq 4$ in the string spectrum at $\lambda = 0$.

We start by looking at the SYM side. From table 5 we see that all letters, and therefore all words at $\lambda = 0$, satisfy the bound

$$\Delta \geq j + \bar{\jmath} + k + p + q\,. \tag{5.2}$$

BPS multiplets start with operators saturating this bound with $j = \bar{\jmath} = 0$. For operators with $j = \bar{\jmath} = 0$, the effective bound $\Delta \geq p + 2q$ is only saturated by the letters ϕ^i and therefore highest weight states of BPS multiplets are built only out of ϕ's. There are two kinds of BPS multiplets in the SYM spectrum:[14] $\frac{1}{2}$-BPS multiplets associated to

[14] With a slight abuse of terminology, some semishort multiplets are sometimes referred to as 1/8 BPS.

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	Δ	SU(4)	$(j,\bar{\jmath})$
	2	[020]	$(0,0)$
$\mathcal{BB}^{\frac{1}{2},\frac{1}{2}}_{[020](0,0)^*}$	3	[002] + [200]	$(0,0)$
		[010]	$(0,1)+(1,0)$
		[101]	$(\frac{1}{2},\frac{1}{2})^*$
	4	$2\cdot$[000]	$(0,0)$
		[000]	$(1,1)^*$
	3	[030]	$(0,0)$
$\mathcal{BB}^{\frac{1}{2},\frac{1}{2}}_{[030](0,0)}$	4	[012] + [210]	$(0,0)$
		[020]	$(0,1)+(1,0)$
		[111]	$(\frac{1}{2},\frac{1}{2})$
$\mathcal{BB}^{\frac{1}{2},\frac{1}{2}}_{[040](0,0)}$	4	[040]	$(0,0)$

Table 8: Bosonic modes with $\Delta \leq 4$ in the supergravity spectrum.

the familiar CPO family $\mathcal{B}^{\frac{1}{2},\frac{1}{2}}_{[0,n,0](0,0)}$ and $\frac{1}{4}$-BPS multiplets $\mathcal{B}^{\frac{1}{4},\frac{1}{4}}_{[q+2,p,q+2](0,0)}$ appearing in the decomposition of long multiplets $\mathcal{A}^{2+p+2q}_{[qpq](0,0)}$ (see appendix). This implies that we can count "nearly protected" long multiplets starting with a scalar HWS by simply counting would-be $\frac{1}{4}$-BPS multiplets [39]. The $\frac{1}{2}$-BPS multiplets are dual to KK descendants of type-IIB supergravity on $\mathrm{AdS}_5 \times S^5$ and appear with degeneracy one at each n. A systematic study of $\frac{1}{4}$-BPS states is given in [49] and a complete list of multiplicities for $\frac{1}{4}$-BPS states with $\Delta \leq 12$ can be found in table 2 of [39].

Let us now consider the string dual picture. To start with we consider long multiplets with $k = q = 0$ i.e. $\mathcal{A}^{\Delta_0}_{[0,\Delta_0-2,0](0,0)}$. These are the so called "BMN multiplets" extensively studied in [54]. Despite their name, the definition of "BMN multiplets" does not necessary require any BMN limit and they are in representations of the AdS supergroup rather than that of the pp-wave geometry [6]. The name is motivated by the fact, proven in [54], that in the large J limit these multiplets contain all BMN operators with two impurities. In the dual SYM theory, they correspond to multiplets starting with operators of the type $\mathrm{Tr}\,\phi^k\,\phi^{(i_1}\ldots\phi_k\,\phi^{i_{\Delta_0-2})}$ [54]. There are $\left[\frac{\Delta_0}{2}\right]$ operators of this type corresponding to all possible distances between the two ϕ_k's (see also [49]). This matches the multiplicity of highest weight states in this representation in the string side (3.13). This can be seen by noticing that there is a single highest weight state in (3.13) with quantum numbers $[0,2\ell+n-2,0]^{2\ell+n}_{(0,0)}$ for any pair (ℓ,n). The multiplicity is then given by all possible choices of ℓ and n for a fixed $\Delta = 2\ell+n$, which is precisely $\left[\frac{\Delta}{2}\right]$.

Another interesting class of nearly protected long multiplets in the string spectrum is the one that includes the highest spin components $\mathcal{A}^{2\ell}_{[000](\ell-1,\ell-1)}$ in (3.13). As we have seen in the previous section, states in these multiplets represent the higher spin conserved currents of $hs(2,2|4)$ and their KK descendants. Once again it is easy in principle although quite laborious in practice to construct the corresponding SYM duals $\mathrm{Tr}\,\phi^i\,\partial_{\mu_1}\ldots\partial_{\mu_{2\ell-2}}\,\phi_i + \ldots$, where $\ldots$ stands for fermion and gauge bilinear needed for 'superprimarity'. There is a single states of this kind for each even spin $s = 2\ell-2$ (the other being superdescendants !) in agreement with the string result and therefore the same higher spin algebra is realized in the two sides of the holographic map.

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More general $\frac{1}{4}$-BPS multiplets $\mathcal{BB}^{\frac{1}{4},\frac{1}{4}}_{[q+2,p,q+2](0,0)}$ appear for example in the decomposition of long multiplets $\mathcal{A}^{2+p+2q}_{[qpq](0,0)}$ in the string spectrum:

$$\mathcal{H}_\ell = \sum_{n=0} [0n0]_{(0,0)} \times ([0,\ell-1,0]_{(0,0)})^2 \times \hat{T}_1^{(2)} + \cdots . \tag{5.3}$$

Unlike the case $q = 0$ ("two impurities" case) discussed above, (5.3) cannot be the only source for $\frac{1}{4}$-BPS states as can be seen by comparing the output of (5.3) to the SYM $\frac{1}{4}$-BPS multiplicities listed in table 2 of [39]. The mismatches start at $\Delta = 8$, indicating that states other than of the type (3.14) show up at this level, cf. appendix B. Reversing the argument, one can try to use the SYM results to get some hints on how our mass formula should be modified in order to account for such states. On the SYM side, the enumeration of $\frac{1}{4}$-BPS states is greatly simplified by the aid of Polya theory. We plan to come back to this issue in the near future.

5.3 Superprimaries and Eratostene's super-sieve

Clearly the comparison of the spectrum of SYM operators and string states can be simplified if we organize SYM operators in supermultiplets. $\mathcal{N} = 4$ supersymmetry transformations of (constant) parameters $\eta^A_\alpha, \bar{\eta}^{\dot{\alpha}}_A$ read

$$\begin{aligned}
\delta_Q(\eta)\phi^i &= \bar{\tau}^i_{AB}\eta^A_\alpha\lambda^{B\alpha} , \\
\delta_Q(\eta)\lambda^A_\alpha &= \frac{1}{2}F_{\mu\nu}\sigma^{\mu\nu}{}_\alpha{}^\beta\eta^A_\beta + \frac{1}{2}g_{\rm YM}[\phi^i,\phi^j]\tau_{ij}{}^A{}_B\eta^B_\alpha , \\
\delta_Q(\eta)\bar{\lambda}_{A\dot{\alpha}} &= \bar{\tau}^i_{AB}\eta^{B\alpha}\sigma^\mu_{\alpha\dot{\alpha}}D_\mu\phi_i , \\
\delta_Q(\eta)F^{\mu\nu} &= i\eta^{A\alpha}\sigma^{[\mu}_{\alpha\dot{\alpha}}D^{\nu]}\bar{\lambda}^{\dot{\alpha}}_A ,
\end{aligned} \tag{5.4}$$

where $\tau^{AB}_i, \bar{\tau}^i_{AB}$ are 4×4 Weyl blocks of Dirac matrices in $D = 6$, and D_μ denote covariant derivatives.

In addition, $\mathcal{N} = 4$ SYM theory is invariant under superconformal transformations that simply read

$$\begin{aligned}
\delta_S(\xi)\phi^i &= \bar{\delta}_Q(\bar{\zeta})\phi^i , \\
\delta_S(\xi)\lambda^A_\alpha &= \bar{\delta}_Q(\bar{\zeta})\lambda^A_\alpha + \phi^i\tau^{AB}_i\xi_{B\alpha} , \\
\delta_S(\xi)\bar{\lambda}_{A\dot{\alpha}} &= \bar{\delta}_Q(\bar{\zeta})\bar{\lambda}_{A\dot{\alpha}} , \\
\delta_S(\xi)F_{\mu\nu} &= \bar{\delta}_Q(\bar{\zeta})F_{\mu\nu} + \xi_A\sigma_{\mu\nu}\lambda^A .
\end{aligned} \tag{5.5}$$

As indicated the 'orbital' part is a susy transformation (5.4) with x-dependent parameters $\zeta^A_\alpha = x_{\alpha\dot{\alpha}}\bar{\xi}^{A\dot{\alpha}}$, $\bar{\zeta}^{\dot{\alpha}}_A = x^{\dot{\alpha}\alpha}\xi_{A\alpha}$.

Superconformal primaries, i.e. HWS of SU$(2,2|4)$ supermultiplets, are defined by the condition

$$\hat{\delta}_S\mathcal{O} \equiv [\xi_A S^A + \bar{\xi}^A\bar{S}_A, \mathcal{O}] = 0 . \tag{5.6}$$

at the origin $x = 0$, where the orbital part $\delta_Q(\zeta) = \bar{\delta}_Q(\bar{\zeta}) = 0$ can be neglected.

From (5.5) one may naively conclude that operators made with only scalar fields be superprimaries. Indeed, this is true at $\lambda = 0$ but the presence of the scalar commutator term in the supersymmetry transformation of the gaugino suggests that only completely symmetric (not necessarily traceless) combinations of scalar fields should remain HWS of (long) SYM multiplets at $\lambda \neq 0$. When interactions are turned on the resolution of the resulting operator mixing and the identification of the renormalized HWS requires in general a rather laborious analysis [26, 27]. In some cases, the problem of diagonalizing the dilation operator [55] can be somewhat simplified or at least systematized by mapping it into the equivalent problem of diagonalizing the hamiltonian of an integrable SO(6) or even SU(2) spin-chain [56, 39]. A supersymmetric version of the spin-chains of [56, 39] should be needed in order to describe fermion and vector impurities.

In principle, at $\lambda = 0$, the condition (5.6) completely characterizes the HWS and is straightforward to impose using (5.5) and their generalization taking care of the presence of derivatives

$$\begin{aligned} \delta_{0S}(\xi)\partial_{\mu_1}\ldots\partial_{\mu_k}\phi_i &= \tau_i^{AB}\xi_A\sigma_{(\mu_1}\partial_{\mu_2}\ldots\partial_{\mu_k)}\bar{\lambda}_B\,, \\ \delta_{0S}\partial_{\mu_1}\ldots\partial_{\mu_k}\lambda^A &= \tau_i^{AB}\xi_B\left(\partial_{(\mu_1} - \frac{1}{4}\sigma_{(\mu_1}\not{\partial}\right)\ldots\partial_{\mu_k)}\phi^i\,, \\ \bar{\delta}_{0S}\partial_{\mu_1}\ldots\partial_{\mu_k}\bar{\lambda}_A &= \xi_A\partial_{(\mu_1}\ldots\partial_{\mu_{k-1}}F_{\mu_k)\nu}\sigma^\nu\,, \\ \delta_{0S}\partial_{\mu_1}\ldots\partial_{\mu_k}F_{\mu\nu} &= \xi_A\partial_{(\mu_1}\ldots\partial_{\mu_k}\sigma_{\mu)\nu}\lambda^A\,. \end{aligned} \tag{5.7}$$

In practice the situation even for 'words' with few 'letters' soon tends to be very intricate. Decomposing the action of the superconformal charges S^A, $\bar{S}_A$ on a given combination of operators (with the same quantum numbers) into irreducible representations of $\mathrm{SO}(4) \times \mathrm{SU}(4) \subset \mathrm{SU}(2,2|4)$ turns out to be very helpful in practical computations. Proceeding this way we have been able to identify the HWS states of the supermultiplets containing operators with $\Delta_0 \leq 4$ viz.

$$\begin{aligned} \mathcal{Z}_{\Delta_0\leq 4} \in \sum_{n=2}^{4}\mathcal{BB}^{\frac{1}{2},\frac{1}{2}}_{[0n0](00)} + \mathcal{CC}^{1,1}_{[000](00)^*} + \mathcal{BC}^{\frac{1}{4},\frac{3}{4}}_{[200](00)} + \mathcal{CB}^{\frac{3}{4},\frac{1}{4}}_{[002](00)} + \mathcal{BB}^{\frac{1}{4},\frac{1}{4}}_{[202](00)} + \\ + \mathcal{CC}^{\frac{1}{2},\frac{1}{2}}_{[010](00)} + \mathcal{BC}^{\frac{1}{4},\frac{1}{2}}_{[210](00)} + \mathcal{CB}^{\frac{1}{2},\frac{1}{4}}_{[012](00)} + \mathcal{CC}^{\frac{1}{2},\frac{1}{2}}_{[020](00)} + \mathcal{CC}^{1,1}_{[000](11)^*} + \mathcal{CC}^{\frac{1}{2},\frac{1}{2}}_{[020](00)} + \\ + \mathcal{CC}^{\frac{1}{4},\frac{1}{4}}_{[101](00)} + 2\,\mathcal{CC}^{\frac{1}{2},\frac{1}{2}}_{[010](\frac{1}{2}\frac{1}{2})} + \mathcal{AC}^{1}_{[000](01)} + \mathcal{CA}^{1}_{[000](10)} + 2\,\mathcal{A}^{4}_{[000](00)}\,. \end{aligned} \tag{5.8}$$

The nature of the multiplets to which the primaries belong is unambiguosly determined by their scaling bare dimension Δ_0. Alternatively one can achieve the same goal by filtering the spectrum by a sort of 'Eratostene's super-sieve'. In other words, starting from the lowest states $[020]^2_{(00)}$ and $[000]^2_{(00)}$, one can subtract from $\mathcal{Z}_{\mathrm{SYM}}$ (4.4) the contributions of the full supermultiplets built on top of them and proceed to HWS of increasing dimension. Proceeding this way one can identify — one after the other — all HWS as lowest dimension states which are not (super)descendants of other HWS and decompose $\mathcal{Z}_{\mathrm{SYM}}$ in supermultiplets of $\mathrm{SU}(2,2|4)$. As a bonus of this 'Eratostene's super-sieve' procedure one should be able to recognize the pairing with Goldstone multiplets of multiplets saturating the unitary bound in free theory. The absence of the required Goldstone multiplets would be an uncontroversial indication that a multiplet remain semishort or 1/4 BPS at large N. Mixing with multi-trace operators may complicate the situation at finite N.

6. Conclusions

Using 'standard' KK techniques and making some plausible assumptions we have derived the spectrum of KK descendants of fundamental string excitations on $\mathrm{AdS}_5 \times S^5$. The results can be written in the deceivingly simple and suggestive form

$$\mathcal{H}_\ell = \sum_{n,\ell} [0n0]_{(0,0)} \times \hat{\mathcal{H}}_\ell^{\text{flat}} , \tag{6.1}$$

with $\hat{\mathcal{H}}_\ell^{\text{flat}}$ the on-shell flat spacetime string spectrum at level ℓ, properly rearranged in representations of the AdS supergroup $\mathrm{SU}(2,2|4)$. Already the fact that on-shell string spectrum in flat spacetime can be organized in multiplets of $\mathrm{SU}(2,2|4)$ is remarkable. In particular a vector of the SO(9) massive little Lorentz group is naturally lifted to an $\mathrm{SO}(1,9)$ vector minus a scalar unphysical component (3.11). The latter accommodates the $\mathrm{SO}(1,3) \times \mathrm{SO}(6)$ quantum numbers displayed in (6.1). Remarkably, states with negative multiplicities have been shown to disappear once the full KK tower of descendants is taken into account.

The result (6.1) generalizes the familiar tower of $\frac{1}{2}$-BPS supergravity multiplets at $\ell = 0$, decorating it with infinite towers of long multiplets $\ell \geq 1$ with increasing spins and masses. Whatever string theory on $\mathrm{AdS}_5 \times S^5$ is, it should contain the spectrum of string theory in flat spacetime at the bottom of its KK tower and the right supergravity content. The spectrum (6.1) satisfies both of these requirements.

Naive extrapolation to small radius combined with the tight constraints imposed by the restoration of HS symmetry led us to conjecture a very simple yet rather effective mass formula that encompasses a large fraction of string excitations. More precisely, bare conformal dimensions in the bulk string theory are assigned in such a way that the $\lambda \to 0$ limit in the boundary theory corresponds to the point where masses of most string states saturate the respective unitary bounds. Based on the resulting mass formula, we confirm that in this limit the spectrum contains precisely one massless HS multiplet in the first Regge trajectory [8] together with its KK recurrences and Goldstone multiplets in the second and higher Regge trajectories. Genuinely massive HS multiplets also appear.

Using Polya enumerative theory and generalizing the results of A. Polyakov [42] we have computed the partition function of gauge invariant single-trace composite operators in free $\mathcal{N} = 4$ SYM theory.[15] The SYM spectrum at $\lambda = 0$ is tested in some detail against the string results. In particular, the SYM spectrum of multiplicities and quantum numbers of higher spin currents and that of marginal and relevant operators are shown to precisely match with the one of string theory based on our simple mass formula. This remarkable agreement may however be slightly deceiving since we expect the need for an extension of the mass formula, even in the limit under consideration, in order to accomodate states which are far from any unitary bound [56]. Finally, we have discussed how to exploit superconformal symmetry in order to select HWS at $\lambda = 0$. Alternatively, an easy-to-apply procedure to achieve the same goal has been described that resembles Eratostene's sieve for prime numbers.

[15] Mixing with multi-trace operators is suppressed in the large N limit relevant to our analysis.

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The string/SYM spectrum displays rich patterns of shortenings of the type extensively studied in [57, 28]. The bulk counterpart of this is a "grande bouffe", i.e. an extended Higgs mechanism, whereby infinitely many higher spin gauge fields eat lower spin fields and become massive, the simplest (non generic) instance being represented by the long Konishi multiplet. Notice that these Goldstone particles are not included in most of the studied HS theories [11, 12, 15, 8] that are restricted to the 'massless' sector that can only capture physics at $\lambda = 0$. Infinitely many Regge trajectories are needed that correspond to massive HS multiplets whose interaction could still be tightly constrained by HS symmetry, as witnessed by the simplicity of the anomalous dimension of HS currents, that calls for a number theoretic origin. Consistent coupling to the dilaton should determine the mass shifts that are holographically dual to the anomalous dimensions.

Although our results are rather encouraging, we expect the need of a refinement of our mass formula, even at $\lambda = 0$, for states belonging to higher Regge trajectories dual to higher twist operators. In general operators with more than two impurities are rather poorly understood, except possibly for those with largest possible anomalous dimension [56]. A very powerful tool for a systematic analysis at non-vanishing coupling is provided by the integrable spin chain proposed in [56] and further exploited in [39], where the spectrum of the dilatation operator [55] for two-impurity operators in the BMN limit [7] is shown to precisely match the spectrum of the light-cone Hamilatonian in the pp-wave geometry [7]. It would be nice to extend these results to "more than two" scalar and to fermionic impurities, the latter presumably in terms of a supersymmetric spin chain. We believe the results presented here motivate further investigation, possibly in relation to other approaches (bits, tensionless strings, etc.) [33]–[37].

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A. Group theory formulas

SU(4) **product formulas:**

$$\begin{aligned}
\mathbf{6} \times [k,p,q] &= [k-1,p,q+1] + [k-1,p+1,q-1] + [k,p+1,q] + \\
&\quad + [k+1,p,q-1] + [k+1,p-1,q+1] + [k,p-1,q]\,, \\
\mathbf{4_s} \times [k,p,q] &= [k-1,p+1,q] + [k,p-1,q+1] + [k,p,q-1] + [k+1,p,q]\,, \\
\mathbf{4_c} \times [k,p,q] &= [k+1,p-1,q] + [k,p+1,q-1] + [k,p,q+1] + [k-1,p,q]\,. \qquad \text{(A.1)}
\end{aligned}$$

Dimension formula:

$$d([k,p,q]) = \frac{1}{12}(k+p+q+3)(k+p+2)(p+q+2)(k+1)(p+1)(q+1)\,. \qquad \text{(A.2)}$$

B. SO(9) content of the flat space string spectrum

$$
\begin{aligned}
\mathrm{vac}_1 &= [0,0,0,0] = \mathbf{1} \\
\mathrm{vac}_2 &= [1,0,0,0] = \mathbf{9} \\
\mathrm{vac}_3 &= [0,0,0,1] + [2,0,0,0] = \mathbf{16} + \mathbf{44} \\
\mathrm{vac}_4 &= [0,1,0,0] + [1,0,0,0] + [1,0,0,1] + [3,0,0,0] \\
&= \mathbf{36} + \mathbf{9} + \mathbf{128} + \mathbf{156} \\
\mathrm{vac}_5 &= [0,0,0,0] + [0,0,0,1] + [0,0,1,0] + [0,1,0,0] + [1,0,0,1] + \\
&\quad + [1,1,0,0] + [2,0,0,0] + [2,0,0,1] + [4,0,0,0] \\
&= \mathbf{1} + \mathbf{16} + \mathbf{84} + \mathbf{36} + \mathbf{128} + \mathbf{231} + \mathbf{44} + \mathbf{576} + \mathbf{450} \\
\mathrm{vac}_6 &= 2{\cdot}[0,0,0,1] + [0,0,0,2] + [0,1,0,0] + [0,1,0,1] + 2{\cdot}[1,0,0,0] + \\
&\quad + 2{\cdot}[1,0,0,1] + [1,0,1,0] + 2{\cdot}[1,1,0,0] + [2,0,0,0] + [2,0,0,1] + \\
&\quad + [2,1,0,0] + [3,0,0,0] + [3,0,0,1] + [5,0,0,0] \\
&= 2{\cdot}\mathbf{16} + \mathbf{126} + \mathbf{36} + \mathbf{432} + 2{\cdot}\mathbf{9} + 2{\cdot}\mathbf{128} + \mathbf{594} + 2{\cdot}\mathbf{231} + \\
&\quad + \mathbf{44} + \mathbf{576} + \mathbf{910} + \mathbf{156} + \mathbf{1920} + \mathbf{1122} \\
\mathrm{vac}_7 &= 2{\cdot}[0,0,0,0] + 2{\cdot}[0,0,0,1] + [0,0,0,2] + 3{\cdot}[0,0,1,0] + 2{\cdot}[0,1,0,0] + \\
&\quad + 2{\cdot}[0,1,0,1] + [0,2,0,0] + 2{\cdot}[1,0,0,0] + 4{\cdot}[1,0,0,1] + [1,0,0,2] + \\
&\quad + [1,0,1,0] + 2{\cdot}[1,1,0,0] + [1,1,0,1] + 3{\cdot}[2,0,0,0] + 3{\cdot}[2,0,0,1] + \\
&\quad + [2,0,1,0] + 2{\cdot}[2,1,0,0] + [3,0,0,0] + [3,0,0,1] + [3,1,0,0] + \\
&\quad + [4,0,0,0] + [4,0,0,1] + [6,0,0,0] \\
&= 2{\cdot}\mathbf{1} + 2{\cdot}\mathbf{16} + \mathbf{126} + 3{\cdot}\mathbf{84} + 2{\cdot}\mathbf{36} + 2{\cdot}\mathbf{432} + \mathbf{495} + 2{\cdot}\mathbf{9} + 4{\cdot}\mathbf{128} + \\
&\quad + \mathbf{924} + \mathbf{594} + 2{\cdot}\mathbf{231} + \mathbf{2560} + 3{\cdot}\mathbf{44} + 3{\cdot}\mathbf{576} + \mathbf{2457} + 2{\cdot}\mathbf{910} + \\
&\quad + \mathbf{156} + \mathbf{1920} + \mathbf{2772} + \mathbf{450} + \mathbf{5280} + \mathbf{2508}\,.
\end{aligned}
\tag{B.1}
$$

C. Decomposition of long multiplets

Let us briefly describe the $\mathcal{N}=4$ superconformal multiplet structure following [28]. The long supermultiplet

$$
\mathcal{A}^{\Delta_0}_{[k,p,q](j,\bar{\jmath})} \equiv T_1 \times [k,p,q]_{(j,\bar{\jmath})}\,, \tag{C.1}
$$

is obtained by the unconstrained action of all sixteen supercharges on the state $[k,p,q]_{(j,\bar{\jmath})}$ of lowest conformal dimension Δ_0. The precise representation content may be found from evaluating the tensor product (C.1), or explicitly by using the Racah-Speiser algorithm as

$$
\mathcal{A}^{\Delta_0}_{[k,p,q](j,\bar{\jmath})} = \sum_{\epsilon_{i\alpha},\bar{\epsilon}^i_{\dot{\alpha}} \in \{0,1\}} \left[(k,p,q)_{(j,\bar{\jmath})} + \epsilon_{i\alpha} q^{i\alpha} + \bar{\epsilon}^i_{\dot{\alpha}} \bar{q}^{\dot{\alpha}}_i\right]\,, \tag{C.2}
$$

type	fraction of susy	HWS	with conformal dimension Δ_0	subset of weights
$\mathcal{B}$	$s=\frac{1}{4}$	$[k,p,q]_{(0,\bar{\jmath})}$	$\frac{1}{2}(3k+2p+q)$	$q_{\mathcal{B}}=q^{\pm}_{2,3,4}$
$\mathcal{B}$	$s=\frac{1}{2}$	$[0,p,q]_{(0,\bar{\jmath})}$	$\frac{1}{2}(2p+q)$	$q_{\mathcal{B}}=q^{\pm}_{3,4}$
$\mathcal{B}$	$s=\frac{3}{4}$	$[0,0,q]_{(0,\bar{\jmath})}$	$\frac{1}{2}q$	$q_{\mathcal{B}}=q^{\pm}_{4}$
$\mathcal{B}$	$s=1$	$[0,0,0]_{(0,\bar{\jmath})}$	0	$q_{\mathcal{B}}=\emptyset$
$\mathcal{C}$	$t=\frac{1}{4}$	$[k,p,q]_{(j,\bar{\jmath})}$	$2+2j+\frac{1}{2}(3k+2p+q)$	$q_{\mathcal{C}}=q^{+}_{1},q^{\pm}_{2,3,4}$
$\mathcal{C}$	$t=\frac{1}{2}$	$[0,p,q]_{(j,\bar{\jmath})}$	$2+2j+\frac{1}{2}(2p+q)$	$q_{\mathcal{C}}=q^{+}_{1,2},q^{\pm}_{3,4}$
$\mathcal{C}$	$t=\frac{3}{4}$	$[0,0,q]_{(j,\bar{\jmath})}$	$2+2j+\frac{1}{2}q$	$q_{\mathcal{C}}=q^{+}_{1,2,3},q^{\pm}_{4}$
$\mathcal{C}$	$t=1$	$[0,0,0]_{(j,\bar{\jmath})}$	$2+2j$	$q_{\mathcal{C}}=q^{+}_{1,2,3,4}$

Table 9: BPS shortening ($\mathcal{B}$) and semi-shortening ($\mathcal{C}$) conditions among the $q^{i\pm}$. Similar relations hold for the weights $\bar{q}^{\pm}_{i}$ of the conjugate supercharges with $[k,p,q]_{(j,\bar{\jmath})}\leftrightarrow[q,p,k]_{(\bar{\jmath},j)}$.

with the sum running over the 2^{16} combinations of the 16 weights $q^{i\pm}$, $\bar{q}^{\pm}_{i}$, $i=1,\ldots 4$ of the supersymmetry charges

$$\begin{aligned}
q^{1\pm} &= [1,0,0]_{(\pm\frac{1}{2},0)}\,, & q^{2\pm} &= [-1,1,0]_{(\pm\frac{1}{2},0)}\,,\\
q^{3\pm} &= [0,-1,1]_{(\pm\frac{1}{2},0)}\,, & q^{4\pm} &= [0,0,-1]_{(\pm\frac{1}{2},0)}\,,\\
\bar{q}^{\pm}_{1} &= [0,0,1]_{(0,\pm\frac{1}{2})}\,, & \bar{q}^{\pm}_{2} &= [0,1,-1]_{(0,\pm\frac{1}{2})}\,,\\
\bar{q}^{\pm}_{3} &= [1,-1,0]_{(0,\pm\frac{1}{2})}\,, & \bar{q}^{\pm}_{4} &= [-1,0,0]_{(0,\pm\frac{1}{2})}\,.
\end{aligned} \tag{C.3}$$

Every q, $\bar{q}$ raises the conformal dimension by $1/2$. In order to make sense out of (C.2) also for small values of $k,p,q,j,\bar{\jmath}$, we need to count representations with negative weights according to the identifications

$$\begin{aligned}
[k,p,q]_{(j,\bar{\jmath})} &= -[-k-2,p+k+1,q]_{(j,\bar{\jmath})} = -[k,p+q+1,-q-2]_{(j,\bar{\jmath})}\\
&= -[k+p+1,-p-2,q+p+1]_{(j,\bar{\jmath})}\,,\\
&= -[k,p,q]_{(-j-1,\bar{\jmath})} = -[k,p,q]_{(j,-\bar{\jmath}-1)}\,.
\end{aligned} \tag{C.4}$$

In particular, this implies that $[k,p,q]_{(j,\bar{\jmath})}$ is counted as zero whenever any of the weights k, p, q equals -1 or one of the spins j, $\bar{\jmath}$ equals $-1/2$.

Short and semi-short multiplets are obtained by acting on the ground state only with a limited number of supercharges, i.e. the sum in (C.2) is restricted to a sum over partial subsets of weights $q^{i\pm}$'s, $\bar{q}^{\pm}_{i}$'s. This requires certain conditions on the ground state $[k,p,q]_{(j,\bar{\jmath})}$ and its conformal dimension Δ_0. In table 9 we have collected the possible subsets of $q^{i\pm}$'s together with the corresponding restrictions on the ground state. Analogous conditions hold for the $\bar{q}^{\pm}_{i}$'s. The general multiplet is obtained by combining a subset of $q^{i\pm}$'s with a subset of $\bar{q}^{\pm}_{i}$'s provided the ground state $[k,p,q]_{(j,\bar{\jmath})}$ satisfies both constraints. Corresponding to the type of shortening and the fraction of preserved supersymmetry, we denote them by $\mathcal{BB}^{s,\bar{s}}$, $\mathcal{BC}^{s,\bar{t}}$, etc.

E.g. the BPS multiplets are for instance $\mathcal{BB}^{\frac{1}{2},\frac{1}{2}}_{[0,n,0](0,0)}$, and $\mathcal{BB}^{\frac{1}{4},\frac{1}{4}}_{[q,p,q](0,0)}$ with lowest conformal dimension $\Delta_0=n,p+2q$, respectively. The former are realized in the KK tower of the $\ell=0$ supergravity sector while the latter appear typically in the decomposition (C.8) of the so called "BMN multiplets" $\mathcal{A}^{p+2q-2}_{[q-2,p,q-2]}$. The KK supergravity multiplets are given in table 2 below with total dimension

$$\dim \mathcal{BB}^{\frac{1}{2},\frac{1}{2}}_{[0,n,0](0,0)} = 2^8 \dim [0, n-2, 0] \,. \tag{C.5}$$

In particular, for $n = 2$ this is the massless supergravity multiplet of dimension 256. Strictly speaking, massless states transform only under the diagonal SU(2) subgroup defining the helicity rotations. We will indicate this by a superscript '*', e.g. $(j, \bar{\jmath}) = (\frac{1}{2}, \frac{1}{2})^*$ denotes a massless state of helicity 1. This is important for the correct counting of degrees of freedom. Another interesting multiplet it the semishort massless multiplet $\mathcal{CC}^{1,1}_{[0,0,0](\ell-1,\ell-1)^*}$ with lowest conformal dimension $\Delta_0 = 2\ell$, which we give in table 4. Its total dimension is given by $256\,(4\ell + 1)$.

Consider now the generic long multiplet $\mathcal{A}^{\Delta_0}_{[k,p,q](j,\bar{\jmath})}$. Unitarity requires the bounds [58]

$$\Delta_0 \geq 2 + 2j + \frac{1}{2}(3k + 2p + q) \,, \qquad \Delta_0 \geq 2 + 2\bar{\jmath} + \frac{1}{2}(3q + 2p + k) \,, \tag{C.6}$$

for the lowest conformal dimension Δ_0 (unless $j = 0$ or $\bar{\jmath} = 0$ and the shortening is of BPS type, in which case also $\Delta = \frac{1}{2}(3k + 2p + q)$ and $\Delta = \frac{1}{2}(3q + 2p + k)$, respectively, are allowed). Simultaneous saturation of the two bounds (C.6) implies $k - q \equiv 2(\bar{\jmath} - j)$ and corresponds to a $\mathcal{CC}$ type shortening of the multiplet. More precisely, the long multiplet then decomposes into [28]

$$\begin{aligned} \mathcal{A}^{2+2j+\frac{1}{2}(3k+2p+q)}_{[k,p,q](j,\bar{\jmath})}\big|_{k-q=2(\bar{\jmath}-j)} \longrightarrow\ & \mathcal{CC}^{\frac{1}{4},\frac{1}{4}}_{[k,p,q](j,\bar{\jmath})} + \mathcal{CC}^{\frac{1}{4},\frac{1}{4}}_{[k+1,p,q](j-\frac{1}{2},\bar{\jmath})} + \\ & + \mathcal{CC}^{\frac{1}{4},\frac{1}{4}}_{[k,p,q+1](j,\bar{\jmath}-\frac{1}{2})} + \mathcal{CC}^{\frac{1}{4},\frac{1}{4}}_{[k+1,p,q+1](j-\frac{1}{2},\bar{\jmath}-\frac{1}{2})} \,. \end{aligned} \tag{C.7}$$

For particular values of $k, p, q, j, \bar{\jmath}$, some of the appearing semi-short multiplets preserve more supersymmetry, and the decomposition becomes[16]

$$\begin{aligned} \mathcal{A}^{2+2j+p}_{[0p0](j,j)} &\longrightarrow \mathcal{CC}^{\frac{1}{2},\frac{1}{2}}_{[0,p,0](j,j)} + \mathcal{CC}^{\frac{1}{4},\frac{1}{2}}_{[1,p,0](j-\frac{1}{2},j)} + \mathcal{CC}^{\frac{1}{2},\frac{1}{4}}_{[0,p,1](j,j-\frac{1}{2})} + \mathcal{CC}^{\frac{1}{4},\frac{1}{4}}_{[1,p,1](j-\frac{1}{2},j-\frac{1}{2})} \,, \\ \mathcal{A}^{2+2j}_{[000](j,j)^*} &\longrightarrow \mathcal{CC}^{1,1}_{[0,0,0](j,j)^*} + \mathcal{CC}^{\frac{1}{4},\frac{3}{4}}_{[1,0,0](j-\frac{1}{2},j)} + \mathcal{CC}^{\frac{3}{4},\frac{1}{4}}_{[0,0,1](j,j-\frac{1}{2})} + \mathcal{CC}^{\frac{1}{4},\frac{1}{4}}_{[1,0,1](j-\frac{1}{2},j-\frac{1}{2})} \,, \\ \mathcal{A}^{2+p+2q}_{[q,p,q](0,0)} &\longrightarrow \mathcal{CC}^{\frac{1}{4},\frac{1}{4}}_{[q,p,q](0,0)} + \mathcal{BC}^{\frac{1}{4},\frac{1}{4}}_{[q+2,p,q](0,0)} + \mathcal{CB}^{\frac{1}{4},\frac{1}{4}}_{[q,p,q+2](0,0)} + \mathcal{BB}^{\frac{1}{4},\frac{1}{4}}_{[q+2,p,q+2](0,0)} \,, \\ \mathcal{A}^{2+p}_{[0p0](0,0)} &\longrightarrow \mathcal{CC}^{\frac{1}{2},\frac{1}{2}}_{[0,p,0](0,0)} + \mathcal{BC}^{\frac{1}{4},\frac{1}{2}}_{[2,p,0](0,0)} + \mathcal{CB}^{\frac{1}{2},\frac{1}{4}}_{[0,p,2](0,0)} + \mathcal{BB}^{\frac{1}{4},\frac{1}{4}}_{[2,p,2](0,0)} \,, \\ \mathcal{A}^{2}_{[000](0,0)^*} &\longrightarrow \mathcal{CC}^{1,1}_{[000](0,0)^*} + \mathcal{BC}^{\frac{1}{4},\frac{3}{4}}_{[200](0,0)} + \mathcal{CB}^{\frac{3}{4},\frac{1}{4}}_{[002](0,0)} + \mathcal{BB}^{\frac{1}{4},\frac{1}{4}}_{[202](0,0)} \,. \end{aligned} \tag{C.8}$$

[16] We note that there is a misprint in the dimension formula (5.49) of [28] which should read

$$\dim \bar{\mathcal{D}}^{\frac{3}{4},\frac{1}{4}}_{[0,0,q](j,0)} = 2^{11} d(0,0,q-1)\left(\mathcal{J} + \frac{3}{2}\right) - 256\left[(2\hat{q}^2 - 1)(\mathcal{J}\underline{+1}) + \hat{q}\right],$$

where $\hat{q} = q + 1$ $\mathcal{J} = 2j + 1$, and $d(k,p,q)$ from (A.2). The missing term in [28] is underlined in the above formula. We checked the correct formula in the decomposition (C.8).

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